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Differentiable Multiobjective Fractional Minimax Programming Duality under Vector η -Convex Functions

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Abstract:
In this paper we introduce the concepts of Differentiable multiobjective fractional Minimax Programming Duality under vector η -convex functions. We establish necessary and sufficient optimality condition and weak duality theorem, strong duality theorem. To develop concept like Duality for multi objective fractional minimax programming problem. To develop some theorems and methods to solve multi objective fractional minimax under vector η -convexity and there by derive duality results for certain generalized multi objective fractional programming problems. To define Vector η -quasi convex and Vector η – Pseudo convex. We establish sufficient optimality condition under vector η – pseudo convexity.

1. Introduction

Mond and Jaya Kumar (2) have introduced the notion of V-invexity for a vector function and discussed its application to a class of constrained multiobjective optimization problems. Bector (1) developed sufficient optimality condition and established duality results under V-invexity under differentiality assumptions.
But no serious attempt made in utilizing the recent develop concept like Duality for multi objective fractional minimax programming problem. Hence in this paper an attempt is made to fill the gap by developing some theorems and methods to solve multi objective fractional minimax under vector η -convexity and there by derive duality results for certain generalized multi objective fractional programming problems.

2. Definition

The following some definitions are used for further discussion.

2.1. Definition

Let R^n denote the n-dimensional Euclidean space and R^n be its non-negative orthant.Let $X_0 \subset R^n$ and

$\frac{fi}{gi} : X_0 \rightarrow R, \; i = 1, 2, \ldots., p,$ be differentiable functions.

A vector function $\frac{fi}{gi} : X_0 \rightarrow R^p$ is said to be Vector η convex if there exist functions.

$\eta : X_0 \text{ x } X_0 \rightarrow R^n$ and $\theta : X_0 \text{ x } X_0 \rightarrow R^+ / \{0\}$ such that for each x, u ∈ X_0 and for i = 1, 2,…, p.

$$\frac{fi\,(x)}{gi\,(x)} - \frac{fi\,(u)}{gi\,(u)} \geq \theta_i\,(x,\; u)\; \nabla\; \frac{fi\,(u)}{gi\,(u)}\; \overline{\eta}\,(x,\; u)$$

for p = 1 and $\overline{\eta}\,(x, u) = \theta_i\,(x, u)\; \eta\,(x, u)$ the above definition reduces to the usual definition of convexity.

2.2. Definition

A vector function $\frac{fi}{gi} : X_0 \rightarrow R^P$ is said to be Vector η – Pseudo convex if there exist function $\eta : X_0 \text{ x } X_0 \rightarrow R^n$ and

$\rho_i : X_0 \text{ x } X_0 \rightarrow R^+ \setminus \{0\}, \; i = 1, 2, \ldots\ldots, \; p$ such that for each x, u ∈ X_0 and
for i = 1, 2,, …. P,

$$\sum_{i=1}^{P} \nabla\; \frac{f_i\;(u)}{g_i\;(u)}\; \eta\;(x,\; u) \geq 0 \Rightarrow \sum_{i=1}^{P} \rho_i\;(x,\; u)\; \frac{f_i\,(x)}{g_i\,(x)} \geq \sum_{i=1}^{P} \rho_i\;(x,\; u)\; \frac{f_i\,(u)}{g_i\,(u)}$$

2.3. Definition

A vector function $\frac{fi}{gi} : X_0 \to R^P$ is said to be

Vector η -quasi convex if there exist functions $\eta : X_0 \times X_0 \to R^n$ and $\phi_i : X_0 \times X_0 \to R^+ / \{0\}$,

i = 1, 2, 3… p such that for each x, u $\in X_0$ and for i = 1, 2, 3, …p

$$\sum_{i=1}^{P} \nabla \frac{f_i(u)}{g_i(u)} \varphi_i(x, u) \le \sum_{i=1}^{P} \varphi_i(x, u) \frac{f_i(u)}{g_i(u)} \Rightarrow \sum_{i=1}^{P} \frac{f_i(u)}{g_i(u)} \eta(x, u) \le 0$$

2.4. Definition

A vector function $\frac{f_i}{g_i} : X_0 \to R^P$ is said to be Vector η- strictly pseudo convex if there exist function $\eta = X_0 \times X_0 \to R^n$ and

$\phi_i : X_0 \times X_0 \to R^+ / \{0\}$, i = 1, 2…, p such that for each x, u $\in X_0$,

$X \neq u$, and for i = 1, 2, .., p,

$$\sum_{i=1}^{P} \varphi_i(x, u) \frac{f_i(x)}{g_i(x)} \le \sum_{i=1}^{P} \varphi_i(x, u) \frac{f_i(u)}{g_i(u)} \Rightarrow \sum_{i=1}^{P} \nabla \frac{f_i(u)}{g_i(u)} \eta(x, u) < 0$$

2.5 Definition

A programming problem (FP) will be called a vector Vector convex programming problem if each of the functions $\frac{f_i}{g_i}$, i = 1, 2, ...p and h_j, j = 1, 2,...m involved in it is a v-convex function.

3. Formulations

3.1. Consider the Following Minimax Programming Problem

(FP) $$\min_{x \in X_0} \max_{1 \le i \le p} \left[\frac{f_i(x)}{g_i(x)} \right]$$

subject to $h_j(x) \le 0$, j = 1, 2, …, m --------- (1)

where $\frac{f_i}{g_i} : X_0 \to R$, i = 1, 2, ..., p and $h_j : X_0 \to R$, j = 1, 2, ...m are differentiable function and $X_0 \subset R^n$ is open.

3.2. The problem (FP) is Equivalent to the Following Problem (EP) Formulated as Follows

(EP) min q

$$\frac{f_i(x)}{g_i(x)} \le q, \; i = 1, 2, ..., p \qquad \text{----------- (2)}$$

$h_j(x) \le 0$, j = 1, 2, …., m ----------- (3)

$x \in X_0$ ----------- (4)

3.3. Dual Problem

Consider the dual (FD) to the equivalent problem (EP) is stated as follows:

(FD) Maximize V

subject to

$$\sum_{i=1}^{P} \lambda_i \nabla \left[\frac{f_i(u)}{g_i(u)} \right] + \sum_{j=1}^{m} \mu_j \nabla h_j(u) = 0 \qquad \text{---------- (5)}$$

$$\lambda_i \left[\frac{f_i(u)}{g_i(u)} - v \right] \ge 0, \; i = 1,2, …., p \qquad \text{---------- (6)}$$

$\mu_j h_j(u) \ge 0$, j = 1, 2, 3,. ,m --------- (7)

$$\sum_{i=1}^{p} \lambda_i = 1 \qquad \text{--------- (8)}$$

$u \in X_0$, $\mu \in R^m_+$, $v \in R$, $\lambda \in R^P_+$, $\lambda \neq 0$ ----------- (9)

4. Necessary and Sufficient Optimality Conditions

4.1. Theorem (Necessary Optimality Condition)

Let $X^0 \in X_0$ be (FP) – optimal with the corresponding (FP) – Optimal value of q^0 , let an appropriate constraint qualification hold for (EP). Then these exist $\lambda^0 \in R$, R_p^+ , $\mu^0 \in R_m^+$ such that (x^0, q^0, λ^0, μ^0) satisfies the following condition.

$$\sum_{i=1}^{P} \lambda_i^0 \nabla \left[\frac{f_i(x^0)}{g_i(x^0)} \right] + \sum_{j=1}^{m} \mu_j^0 \nabla h_j(x^0) = 0 \quad \text{---------- (10)}$$

$$\lambda_i^0 \left[\frac{f_i(x^0)}{g_i(x^0)} - q^0 \right] = 0, \ (i = 1, 2, ...p) \quad \text{---------- (11)}$$

$$\mu_j^0 \ h_j(x^0) = 0, j = 1, 2, \ldots, m, \quad \text{---------- (12)}$$

$$\frac{f_i(x^0)}{g_i(x^0)} \leq q^0 , i = 1, 2, \ldots p \quad \text{----------- (13)}$$

$$h_j(x^0) \leq 0, \in j = 1, 2, \ldots, m \quad \text{----------- (14)}$$

$$\sum_{i=1}^{p} \lambda_i^0 = 1 \quad \text{----------- (15)}$$

Proof: -It follows directly by writing the necessary optimality condition to the problem (EP) $\lambda_i^0 \in R_p^+$, $\mu^0 j \in R_m^+$ such that the following conditions hold.

$$\sum_{i=1}^{p} \lambda i^0 \nabla \left(\frac{fi(x^0)}{gi(x^0)} \right) + \sum_{j=1}^{m} \mu j^0 \nabla hj(x^0) = 0$$

μj^0 hj (x^0) = 0, j = 1, 2, …m,

$\lambda i^0 \geq 0$, i = 1, 2, …. P,

$\mu j^0 \geq 0$, j = 1, 2, ….m,

where $q^0 = \dfrac{f_i(x^0)}{g_i(x^0)}$, i= 1, 2, ….p,

setting

$$\lambda_i^0 = \frac{\lambda i}{\sum_{i=1}^{p} \lambda i}, \ i = 1, 2, ..., p$$

$$\mu_j^0 = \frac{\mu_j}{\sum_{i=1}^{k} \lambda_i}, \ j = 1, 2, ... m$$

We observed that the conditions 10 to 15 hold.

Which completed proof of the theorem.

We now establish sufficient optimality condition under vector η – pseudo convexity.

4.2. Theorem (Sufficient Optimality Condition)

Let (x^0, q^0, λ^0, μ^0) with $x^0 \in X_0$, $q^0 \in R$, $\lambda^0 \in R_p^+$ and $\mu^0 \in R_m^+$ satisfy relations 3.10 to 3.15 at x^0, let

i. $\left(\lambda_1^0 \dfrac{f_1(x)}{g_1(x)}, ... \ \lambda_p^0 \dfrac{fp(x)}{gp(x)} \right)$ be vector η – pseudo convex with respect to η, and

ii. ($\mu_1^0 h_1(x)$, ... $\mu_m^0 h_m(x)$) be vector η quasi convex with respect of η.Then x^0 is (FP) – optimal with the corresponding optimal objective value equal to q^0.

Proof: -Let S and T denote the set of feasible solutions of (FP) and (EP) respectively.

Let $x \in S$ be arbitrary. From (3) and (12) we have

$\mu_j^0 hj(x) \leq \mu_j^0 hj(x^0)$ for j = 1, 2, …, m --------(16)

But the vector function $(\mu_1^0 h_1, \ldots\ldots \mu_m^0 h_m)$ is

vector η quasi convex at $x^0 \in S$. Therefore, there exist functions

$\eta : X_0 \times X_0 \rightarrow R^n$ and $\phi j : X_0 \times X_0 \rightarrow R^+ / \{0\}$, j = 1, 2, …., m such that

$$\sum_{j=1}^{m} \phi_j(x, x^0)\mu_j^0 hj(x) \leq \sum_{j=1}^{m} \phi_j(x, x^0)\mu_j^0 hj(x^0)$$

$$\Rightarrow \quad \sum_{j=1}^{m} \mu_j \nabla hj(x^0)\eta(x, x^0) \leq 0 \qquad \text{------- (17)}$$

since $\phi : X_0 \times X_0 \rightarrow R^+ / \{0\}$, j = 1, 2, ….m, therefore, (16) yields

$$\sum_{j=1}^{m} \varphi_j(x, x^0)\mu_j^0 hj(x) \leq \sum_{j=1}^{m} \varphi_j(x, x^0)\mu_j^0 hj(x^0) \qquad \text{------ (18)}$$

(18) along with (17) yields

$$\sum_{j=1}^{m} \mu_j^0 \nabla hj(x^0)\eta(x, x^0) \leq 0 \qquad \text{--------(19)}$$

10 along with 19 yields

$$\sum_{i=1}^{p} \lambda i^0 \nabla \frac{fi(x^0)}{gi(x^0)}\eta(x, x^0) \geq 0 \qquad \text{----------(20)}$$

(20) along with the fact that the vector function

$\left(\lambda_1^0 \frac{f_1}{g_1}, \ldots\ldots \lambda_p^0 \frac{f_p}{g_p}\right)$ is vector η – pseudo convex at $X^0 \in s$ yields that functions

$\theta i : X_0 \times X_0 \rightarrow R^+ / \{0\}$, i = 1, 2, …. P such that

$$\sum_{i=1}^{p} \theta_i(x, x^0)\lambda_i^0 \frac{fi(x)}{gi(x)} \geq \sum_{i=1}^{p} \theta_i(x, x^0)\lambda_i^0 \frac{fi(x^0)}{gi(x^0)} \qquad \text{------------ (21)}$$

(21) together with (11) gives

$$q \sum_{i=1}^{p} \theta_i(x, x^0)\lambda_i^0 \geq q^0 \sum_{i=1}^{p} \theta_i(x, x^0)\lambda_i^0 \qquad \text{------------- (22)}$$

for $(x, q) \in T$. But

$$\sum_{i=1}^{q} \theta_i(x, x^0)\lambda_i^0 > 0$$

Hence (22) give that for all $(x, q) \in T$

$q \geq q^0$

Hence the proof.

5. Duality Theorems

5.1. Weak Duality Theorem

Let $(x, q) \in T$ and $(u, v, \lambda, \mu) \in$ w.Let w denote the set of all feasible solutions of (FD). Let $\left(\lambda_1 \frac{f_1}{g_1}, \lambda_2 \frac{f_2}{g_2} \ldots, \lambda_p \frac{f_p}{g_p}\right)$ be vector η -pseudo convex with respect to η and $(\mu_1 h_1, \ldots, \mu_m h_m)$ be vector η -quasi convex with respect to η, for all feasible solution of (EP) and (FD). Then

$q \geq v$.

Proof :-since $(x, q) \in T$ and $(u, v, \lambda, \mu) \in W$,

Therefore, we have for j = 1, 2, …, m,

$\mu_j h_j(x) \leq \mu_j h_j(u)$ ---------- (23)

But the vector function $(\mu_1 h_1, \ldots. \mu_m h_m)$ is vector η- quasi convex, therefore, there exist

function $\eta : X_0 \times X_0 \rightarrow R^n$ and $\phi_j : X_0 \times X_0 \rightarrow R^+ / \{0\}$, j = 1, 2, …., m, such that

$$\sum_{j=1}^{m} \varphi_j(x, \mu)\mu_j h_j(x) \leq \sum_{j=1}^{m} \varphi_j(x, \mu)\mu_j h_j(u)$$

$$\Rightarrow \sum_{j=1}^{m} \mu_j \nabla h_j(u)\eta(x,u) \leq 0 \quad \text{---------- (3.24)}$$

$\phi j : X_0 \times X_0 \to R^+ / \{0\}$, j = 1, 2, …., m, therefore (23) yields

$$\sum_{j=1}^{m} \varphi_j\ (x,u)\,\mu_j\, h_j(x) \leq \sum_{j=1}^{m} \varphi_j\ (x,u)\,\mu_j\, h_j(x) \quad \text{--------- (25)}$$

(25) along with (24) yields

$$\sum_{j=1}^{m} \mu_j\ \nabla\, hj(u)\ \eta(x,u) \ \leq 0 \quad \text{--------- (26)}$$

(25) and (26) yield

$$\sum_{j=1}^{p} \lambda_j\ \nabla \left[\frac{fi(u)}{gi\ (u)}\right] \eta(x,u) \geq 0 \quad \text{---------- (27)}$$

(27) along with the fact that the vector function $\left(\lambda_1 \frac{f_1}{g_1}, \ldots \lambda_p \frac{f_p}{g_p}\right)$ is vector η -pseudo convex gives that there exists functions$\theta_i : X_0 \times X_0 \to R^+ / \{0\}$, i = 1, 2, …, p such that

$$\sum_{i=1}^{p} \theta_j\ (x,u)\,\lambda_i \left[\frac{f_i(x)}{g_i(x)}\right] \geq \sum_{i=1}^{p} \theta_j\ (x,u)\,\lambda_i \left[\frac{f_i(u)}{g_i(u)}\right] \quad \text{------------ (28)}$$

(28) together with (2) and (6) gives

$$q \sum_{i=1}^{p} \theta_i\ (x,u)\,\lambda_i \geq V \sum_{i=1}^{p} \theta_i\ (x,u)\,\lambda_i \quad \text{---------- (29)}$$

for all (EP) – feasible and (FD) – feasible solutions.

But $\sum_{j=1}^{p} \theta_i\ (x,u)\lambda i > 0,$ therefore, for all (EP)- feasible and (FD) – feasible solutions.

(29) yields. $q \geq v$

This proves the theorem.

5.2. Strong Duality Theorem

Let $(\overline{x}, \overline{q}) \in T$ be (EP) - optimal at which an appropriate constraint qualification holds.

Then there exist $\overline{\lambda}$ and $\overline{\mu}$ such that $(\overline{x}, \overline{q}, \overline{\lambda}, \overline{\mu})$ is (FD) – feasible and the corresponding objective values of (EP) and (FD) are equal.If also, the hypothesis of theorem 5.1 hold,

then $(\overline{x}, \overline{q}, \overline{\lambda}, \overline{\mu})$ is (FD) – optimal.

Proof: - Since (x^0, q^0) is (EP) – optimal, therefore,
there exists (λ^*, μ^*) such that $(x^0, q^0, \lambda^0, \mu^0)$ is (FD) – optimal.
Also for j = 1, 2, …m, $\mu_j^0\, h_j(x^0) \leq 0$ and $\mu_j^0\, h_j(u) \geq 0$,
therefore, $\mu_j^0\, h_j(x^0) \leq 0 \leq \mu_j^0\, h_j(u)$ for j = 1, 2, …m.

Along the lines of theorem 5.1 vector η- quasi invexity
Of$(\mu_1^0 h_1 \ldots \mu_m^0 h_m)$ with respect to η yields
functions $\phi_j : X_0 \times X_0 \to R^+ / \{0\}$, j = 1, 2, …m, such that

$$\sum_{j=1}^{m} \mu_j^0\, \nabla\ hj\ (\mu^0)\ \eta\ (x^0, u) \ \leq 0 \quad \text{--------- (30)}$$

(30) together with (5) yields

$$\sum_{i=1}^{p} \lambda_i^0\, \nabla \frac{f_i(u^0)}{g_i(u^0)}\ \eta\ (x^0, u^0) = 0 \quad \text{------- (31)}$$

(31) along with the vector η pseudo convexity
of $\lambda_i^0 \frac{fi}{gi}$, i = 1,2 .... p (or) $\left\{\left(\lambda_1^0 \frac{f_1}{g_1}, \ldots\ldots, \lambda_p^0 \frac{f_p}{g_p}\right)\right\}$ with respect of η yields
that $\exists$ functions $\theta_i : X_0 \times X_0 \to R^+ / \{0\}$, i = 1, 2, …p such that

$$\sum_{i=1}^{p} \lambda_i^0 \theta_i (x^0, u^0) \frac{f_i(x^0)}{g_i(x^0)} = \sum_{i=1}^{p} \lambda_i^0 \theta_i (x^0, u^0) \frac{f_i(u^0)}{g_i(u^0)} \quad \text{-------- (32)}$$

since (x^0, q^0) is EP – feasible and $(u^0, v^0, \lambda^0, \mu^0)$ is (FD) – feasible, therefore, using the constrains (2) and (19), we have from (32)

$$q^0 \sum_{i=1}^{p} \lambda i^0 \ \theta i (x^0, u^0) = V^0 \sum_{i=1}^{p} \lambda i^0 \ \theta i (x^0, u^0) \quad \text{--------- (33)}$$

As $\sum_{i=1}^{p} \lambda i^0 \ \theta i (x^0, u^0) = 0,$ therefore, from (33) we have

$q^0 = v^o$.

Hence proved.

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