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Convergence Theorem for Fuzzy Random Variables

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Abstract:
In this paper, we establish some results which are convergence in probability and convergence in distribution for fuzzy random variables. By treating a fuzzy subset as a tower of subsets we introduce the convergence concepts of fuzzy sets. The inner connection of convergence in probability and in distribution for fuzzy random variables is investigated in this paper.

Keywords: *Fuzzyrandom variables, Fuzzy distribution functions, Convergence in probability, Convergence in distribution.*

1. Introduction

Convergence concepts are the foundations of mathematical analysis while the convergence of fuzzy sets is the foundation of fuzzy analysis. Since the introduction of fuzzy sets by Zadeh researchers have been concerned with the calculus of fuzzy functions and generalization of the convergence concepts.

Bounded and convergent sequence of fuzzy numbers were first introduced by Matloka [1]. He also showed that every convergent sequence is bounded. Nurag and Savas [2] introduced and discussed the concept of statistically convergent and statistically canchy sequence of fuzzy numbers. The works of Artshera [3], Mosco [4], Salinetti and Wets [5, 6] and Wiseman [7] trace development of theory and applications of set convergence.

2. Preliminaries

In this section, we describe some basic concepts of fuzzy numbers. Let R denote the real line. A fuzzy number is a fuzzy set$\tilde{u}$: $R \to [0,1]$ with the following properties;

1) $\tilde{u}$ is normal, i.e., there exists $x \epsilon R$ such that $\tilde{u}(x) = 1$.
2) $\tilde{u}$is upper semicontinous.
3) supp$\tilde{u} = cl\{x \epsilon R : \tilde{u}(x) > 0\}$ is compact.

4) $\tilde{u}$ is a convex fuzzy set, i.e.,$\tilde{u}(\lambda x + (1 - \lambda)y) \geq \min(\tilde{u}(x)), \tilde{u}(y))$ for $x, y \epsilon R$ and $\lambda \epsilon [0,1]$.

Let $F(R)$ be the family of all fuzzy numbers. For a fuzzy set $\tilde{u}$, if we define

$$L_\alpha \tilde{u} = \begin{cases} \{x: \tilde{u}(x) \geq \alpha\}, & 0 < \alpha \leq 1, \\ \text{Supp } \tilde{u}\alpha = 0 \end{cases}$$

Then it follows that $\tilde{u}$ is a fuzzy number if and only if $L_1\tilde{u} \neq \phi$ and $L_\alpha \tilde{u}$ is a closed bounded interval for each $\alpha \epsilon [0,1]$. From this characterization of fuzzy numbers, a fuzzy number $\tilde{u}$ is completely determined by the end points of the intervals $L_\alpha \tilde{u} = [u_x^-, u_x^+]$.

- Theorem 2.1.*For*$\tilde{u} \epsilon F(R)$, *denote* $u^-(\alpha) = u_x^-$*and*$u^+(\alpha) = u_x^1$*by considering as functions of*$\alpha \epsilon [0,1]$. *Then*

1) $u^-(\alpha)$*is a bounded increasing function on* [0, 1].
2) $u^+(\alpha)$*is a bounded decreasing function on* [0, 1].
3) $u^-(1) \leq u^+(1)$.
4) $u^-(\alpha)$*and*$u^+(\alpha)$ *are left continuous on* (0, 1] *and right continuous at* 0.
5) *If*$v^-(\alpha)$*and*$v^+(\alpha)$*satisfy above* (1)-(4), *then there exists a unique*$\tilde{v} \epsilon F(R)$*such that*$L_\alpha \tilde{v} = [v^-(\alpha), v^+(\alpha)]$.

The above theorem implies that we can identify a fuzzy number $\breve{u}$with the parametrized representation $\{(u_x^-,\ u_x^+)|0 \leq \alpha \leq 1\}$. Suppose now that $\tilde{u}, \breve{v} \epsilon\ F(R)$are fuzzy numbers represented by $\{(u_x^-, u_x^+)|0 \leq \alpha \leq 1\}$and $\{(v_x^-, v_x^+)\ |\ 0 \leq \alpha \leq 1\}$, respectively, if we define.

$$(\tilde{u} + \tilde{v})(z) = \sup_{x+y=2} \min(\tilde{u}(x), \tilde{v}(\mathrm{y})).$$

$$(\lambda\tilde{u})(z) = \begin{cases} \tilde{u}\left(\frac{z}{\lambda}\right), & \lambda \neq 0, \\ \tilde{0}, & \lambda = 0, \end{cases}$$

Where $\tilde{0} = I_{\{0\}}$is the indicator function of $\{0\}$, then

$$\tilde{u} + \tilde{v} = \{(u_x^- + v_x^-, v_z^+ + v_x^+)|0 \leq \alpha \leq 1\},$$

$$\lambda\tilde{u} = \begin{cases} \{(\lambda u_x^-, \lambda u_x^+)|0 \leq \alpha \leq 1\}, \lambda \geq 0, \\ \{(\lambda u_x^+, \lambda u_x^-)|0 \leq \alpha \leq 1\}, \lambda \geq 0 \end{cases}$$

Now, we define two metrics d, d^* on $F(R)$ by

$$d(\breve{u}\tilde{v}) = \sup_{0 \leq x \leq 1} d_{\mathrm{H}}(L_\alpha\tilde{u}, L_\alpha\tilde{v})(2.1)$$

$$d^*(\breve{u}\tilde{v}) = \int_0^1 d_{\mathrm{H}}(L_\alpha\tilde{u}, L_\alpha\tilde{v})\, \mathrm{d}\alpha, \qquad (2.2)$$

Where $d_{\mathrm{H}}$is the Hausdorff metric defined as

$$d_{\mathrm{H}}(L_\alpha\tilde{u}, L_\alpha\tilde{v}) = \max(|u_\alpha^-\ v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|).$$

Also, the norm $\|\tilde{u}\|$of fuzzy number $\tilde{u}$ will be defined as

$$\|\tilde{u}\| = d\left(\tilde{u}, \tilde{0}\right) = \max(|u_0^-|, |u_0^+|).$$

Then it is well-know that $(F(R), d^*)$ is separable but $(F(R), d)$ is non-separable.

3. Fuzzy Random Variables

Throughout this paper, (Ω, A, P) denotes a complete probability space. If $\tilde{x}: \Omega \longrightarrow F(R)$ is a fuzzy number valued function and $B$is a subset of R, then$\tilde{X}^{-1}(B)$ denotes the fuzzy subset of Ω defined by

$$\tilde{X}^{-1}(B)(\omega) = \sup_{x\ \epsilon\ B} \tilde{X}(\omega)(x)$$

for every $\omega\ \epsilon\ \Omega$. The function $\tilde{X}: \Omega \longrightarrow F(R)$is called a fuzzy random variable if for every closed subset B of R, the fuzzy set $\tilde{X}^{-1}(B)$is measurable when consider as a function from Ω to [0, 1]. If we denote $\tilde{X}(\omega) = \{X_x^-(\omega), X_x^{-1}(\omega)|0 \leq \alpha \leq 1\}$, then it is well-know that $\tilde{X}$ is a fuzzy random variable if and only if for each $\alpha\ \epsilon\ [0,1]$, X_x^- and X_x^+ are random variable in the usual sense (for details, see Ref.[11]). Hence, if $\sigma(\tilde{X})$ is the smallest σ-field which makes $\tilde{X}$ is consistent with $\sigma(\{X_x^-, X_x^+|0 \leq \alpha \leq 1\})$.This enables us to define the concept of independence for fuzzy random variables as in the case of classical random variables.

3.1. Fuzzy Random Variable and Its Distribution Function and Exception

Given a real number x, we can induce a fuzzy number $\tilde{u}$ with member ship function $\xi_{\tilde{x}}(r)$such that $\xi_{\tilde{x}}(x) = 1$ and $\xi_{\tilde{x}}(r) < 1$ for $r \neq x$ (i.e., the membership function has a unique global maximum at x). We call $\tilde{u}$ as a fuzzy real number induced by the real number x.

A set of all fuzzy real numbers induced by the real number system the relation $\sim$ on $\mathcal{F}_{\mathbb{R}}$ as $\tilde{x}^{-1} \sim \tilde{x}^{-2}$ if and only if $\tilde{x}^{-1}$ and $\tilde{x}^{-2}$ are induce same real number x. Then $\sim$ is an equivalence relation, which equivalence classes$[\tilde{x}] = \{\tilde{a}|\tilde{a} \sim \tilde{x}\}$. The quotient set $\mathcal{F}_{\mathfrak{R}}/\sim$ is the equivalence classes. Then the cardinality of $\mathcal{F}_{\mathfrak{R}}/\sim$ is equal to the real number system $\mathbb{R}$ since the map $\mathbb{R} \longrightarrow \mathcal{F}_{\mathfrak{R}}/\sim$ by $x \mapsto [\tilde{x}]$ is Necall $\mathcal{F}_{\mathfrak{R}}/\sim$ as the fuzzy real number system.

Fuzzy real number system $(\mathcal{F}_R/\sim)_R$ consists of canonical fuzzy real number we call $\mathcal{F}_R/\sim)_R$ as the canonical fuzzy real number system. be a measurable space and $\mathbb{R}, \mathcal{B}$ be a Borel measurable space. $\wp(\mathbf{R})$ (power set of **R**) be a set-valued function. According to is called a fuzzy-valued function if $\{(x,y): y\ \epsilon\ f(x)\}$is $\mathcal{M} \times \mathcal{B}$. $f(x)$is called a fuzzy-valued function if $f: X \longrightarrow \mathcal{F}$ (the set of all numbers). If $\tilde{f}$ is a fuzzy-valued function then $\tilde{f}_x$ is a set-valued function [0, 1]. $\tilde{f}$is called (fuzzy-valued) measurable if and only if $\tilde{f}_x$ is (set- urable for all $\alpha\ \epsilon\ [0,1]$.

make fuzzy random variables more tractable mathematically, we strong sense of measurability for fuzzy-valued functions. $\tilde{f}(x)$ is d-fuzzy-valued function if $\tilde{f}: X \longrightarrow \mathcal{F}_A$ (the set of all closed fuzzy-set) $\tilde{f}(x)$ be a closed-fuzzy-valued function defined on X. From Wu wing two statements are equivalent.

$\tilde{f}_\alpha^U(x)$are (real-valued) measurable for all $\alpha\ \epsilon\ [0,1]$.

fuzzy- valued) measurable and one of $\tilde{f}_\alpha^L(x)$ and $\tilde{f}_\alpha^U(x)$ is (real-value) measurable for all $\alpha\ \epsilon\ [0,1]$

A fuzzy random variable called strongly measurable if one of the above two conditions is easy to see that the strong measurability implies measurability. μ)be a measure space and $(\mathbb{R}, \mathcal{B})$ be a Borel measurable space. $\wp(\mathbb{R})$be a set-valued function. For $K \subseteq$ R, the inverse image of f

$$= \{x \in X : f(x) \bigcap K \neq \emptyset\}.$$

u) be a complete σ-finite measure space. From Hiai and Umehaki ing two statements are equivalent:

Borel set $K \subseteq \mathbb{R}, f^{-1}(K)$ is measurable (i.e. $f^{-1}(K) \in \mathcal{M}$), $y \in f(x)\}$ is $\mathcal{M} \times \mathcal{B}$- measurable.

If $\tilde{x}$ is a canonical fuzzy real number then $\tilde{x}_1^{-L} = \tilde{x}_1^U$, Let $\tilde{X}$ be a fuzzy random variable. Then, from Proposition 3.2, $\tilde{X}_x^L$ and $\tilde{X}_x^U$ are random variables for all $x$and$\tilde{x}_1^{-L} = \tilde{x}_1^U$. Let $F(x)$ be a continuous distribution function of a random variable X. Let $\tilde{x}_\alpha^{-L}$and$\tilde{x}_\alpha^U$ have the same distribution function $F(x)$ for all $\alpha \in [0,1]$. For any fuzzy observation $\tilde{x}$ of fuzzy random variable $\tilde{X}\left(\tilde{X}(\omega) = \tilde{x}\right)$, the α-level set $\tilde{x}_\alpha$is$\tilde{x}_\alpha = [\tilde{x}_\alpha^L, \tilde{x}_\alpha^U]$.We can see that $\tilde{x}_\alpha^L$and$\tilde{x}_\alpha^U$ are the observations of $\tilde{x}_\alpha^L$and$\tilde{x}_\alpha^U$, respectively. From Proposition 2.4 $\tilde{X}_\alpha^L(\omega) = \tilde{x}_\alpha^L$ and $\tilde{X}_\alpha^U(\omega) = \tilde{x}_\alpha^U$ are continuous with respect to α for fixed ω. Thus $\tilde{X}_\alpha^L, \tilde{x}_\alpha^U$ is continuously shrinking with respect to α. Since $[\tilde{X}_\alpha^L, \tilde{x}_\alpha^U]$ is the disjoint union of $[\tilde{X}_\alpha^L, \tilde{x}_1^L]$ and $(\tilde{X}_1^U, \tilde{x}_\alpha^U]$ (note that $\tilde{X}_1^L = \tilde{x}_1^U$), for any real number $x \in [\tilde{X}_\alpha^L, \tilde{x}_\alpha^U]$, we have $x = \tilde{x}_\beta^L$ or $F(\tilde{x}_\beta^U)$ with x. If we construct an interval

$$A_\alpha = [\min\left\{\inf_{\alpha\le\beta\le1} F(\tilde{x}_\beta^L), \inf_{\alpha\le\beta\le1} F(\tilde{x}_\beta^L),\right\},$$
$$\max\left\{\sup_{\alpha\le\beta\le1} F(\tilde{x}_\beta^L), \sup_{\alpha\le\beta\le1} F(\tilde{x}_\beta^L)\right\}].$$

then this interval will contain all of the distributions. (values) associated with each of $x \in [\tilde{x}_\alpha^L, \tilde{x}_\alpha^U]$. We denote $\tilde{F}(\tilde{x})$ the fuzzy distribution function of fuzzy random variable $\tilde{X}$. Then we define the membership function of $\tilde{F}(\tilde{x})$ for any fixed $\tilde{x}$ by

$$\xi_{\tilde{F}(\tilde{x})}(r) = \sup_{0\le\alpha\le1} \alpha|_A, (r)$$

via the form of “Resolution Identity” in Proposition 2.6 (i). and we also say that the fuzzy distribution function $\tilde{F}(\tilde{x})$ is induced by the distribution function $F(x)$. Since $F(x)$ is continuous.from Propositions 2.4 and 2.1. we can rewrite A_α as

$$A_\alpha = [\min\left\{\min_{\alpha\le\beta\le1} F(\tilde{x}_\beta^L), \min_{\alpha\le\beta\le1} F(\tilde{x}_\beta^L),\right\},$$
$$\max\left\{\max_{\alpha\le\beta\le1} F(\tilde{x}_\beta^L), \max_{\alpha\le\beta\le1} F(\tilde{x}_\beta^L)\right\}].$$

In order to discuss the convergence in distribution for fuzzy random variables in Section 4, we need to claim that $\tilde{F}(\tilde{x})$ is a closed-fuzzy-valued function. First of all, we need the following proposition,
We shall discus the strong and weak convergence in distribution for fuzzy random variables in this section. From Proposition 3.2, we propose the following definition.

- Definition 3.1Let$\tilde{X}$ and $\{\tilde{x}_n\}$ be fuzzy random variables defined on the same probability space $(\Omega, \mathcal{A}, \mathcal{P})$.

i) We say that $\{\tilde{X}_n\}$ converges in distribution to $\tilde{X}$ level-vise if $(\tilde{x}_n)_\alpha^L$, and $(\tilde{x}_n)_\alpha^U$ converge in distribution to $\tilde{X}_\alpha^L$ and $\tilde{X}_\alpha^U$, respectively for all α. Let $(\tilde{x})$ and $\tilde{F}(\tilde{x})$ be the respective fuzzy distribution functions of $\tilde{X}_\alpha$ and $\tilde{X}$. We say that $\{\tilde{X}_\alpha\}$ converges in distribution to $\tilde{X}$strongly if

$$\lim_{n\to\alpha} \tilde{F}_n(\tilde{x}) \overset{s}{\Rightarrow} \tilde{F}(\tilde{x}).$$

ii) We say that $\{\tilde{X}_n\}$ converges in distribution to $\tilde{X}$ weakly if

$$\lim_{n\to\alpha} \tilde{F}_n(\tilde{x}) \overset{w}{\Rightarrow} \tilde{F}(\tilde{x}).$$

From the uniqueness of convergence in distribution for usual random variables and proposition 2.9 (ii), we conclude that the above three kinds of convergence have the unique limits.

4. Convergence

The following theorem is a fuzzy version of theorem due to slutsky.

- Theorem: 4.1

Let $\{X_n\}$ and $\{Y_n\}$ be sequences of fuzzy random variables. Let $X_n \overset{d}{\to} X$ and $Y_n \overset{P}{\to} C$ (real constant) than

i) $X_n + Y_n \overset{d}{\to} X + C$ (i.e. distribution function of $X_n + Y_n$ tends to F(x-c)) at each continuity point in F.

ii) $X_n Y_n \xrightarrow{d} CX$ (i.e. the distribution function of $X_n Y_n$ tends to F(x/c) at each continuity part of F).

iii) $\frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{C}$ ($C \neq 0$, i.e. the distribution function $X_n Y_n$ tends to F(cx) at each point of continuity of F)

Proof:

(i) Choose and fix x such that $(x - c)$ is a continuity point of F. Let $\epsilon > 0$ be such that $x - c + \epsilon$ and $x - c - \epsilon$ are also continuity points of x.

Then

$$F(x) = P[\bigcup_{\alpha \epsilon (0,1)} \alpha(x_n)_\alpha + \bigcup_{\alpha \epsilon (0,1)} \alpha(y_n)_\alpha \leq x$$

$$= P[\bigcup_{\alpha \epsilon (0,1)} \alpha((x_n)_\alpha + (y_n)_\alpha \leq x]$$

$$= \bigcup_{\alpha\epsilon(0,1)} \alpha[P(x_n)_\alpha (y_n)_\alpha \leq x]$$

$$\leq \bigcup_{\alpha\epsilon(0,1)} \alpha[P(x_n)_\alpha + (y_n)_\alpha \leq x, \quad |(x_n)_\alpha - c| < \epsilon] + \bigcup_{\alpha\epsilon(0,1)} \alpha\, [P\, |(y_n)_\alpha - c| \geq \epsilon]$$

$$\leq \bigcup_{\alpha\epsilon(0,1)} \alpha\, [P(x_n)_\alpha \leq x - (c - \epsilon)] + \bigcup_{\alpha\epsilon(0,1)} \alpha\, [P|(y_n)_\alpha - c| \geq \epsilon]$$

$$\overline{\lim} F_{x_n+y_n}(x) \leq \overline{\lim} P((x_n)_\alpha \leq x - c + \epsilon) + \overline{\lim} P(|(y_n)_\alpha - c| < \epsilon)$$

Since $y_n \xrightarrow{P} c$ then last term $\longrightarrow 0$

and so

$$\overline{\lim} F_{x_n+y_n}(x) \leq \bigcup_{\alpha\epsilon(0,1)} \alpha\, [P(x_n)_\alpha \leq x - (c - \epsilon)]$$

$\therefore\ \overline{\lim} F_{x_n+y_n}(x) \leq F(x - c + \epsilon)$for > 0 . (4.1)

Now

$$\bigcup_{\alpha\epsilon(0,1)} \alpha\ [P(x_n)_\alpha + (y_n)_\alpha > x]$$

$$\leq \bigcup_{\alpha\epsilon(0,1)} \alpha\ [P(x_n)_\alpha + (y_n)_\alpha > x, |(y_n)_\alpha - c| < \epsilon + \bigcup_{\alpha\epsilon(0,1)} \alpha\ P[\, |(y_n)_\alpha - c| \geq \epsilon]$$

$$\leq \bigcup_{\alpha\epsilon(0,1)} \alpha\ [P((x_n)_\alpha + c + \epsilon > x)] + \bigcup_{\alpha\epsilon(0,1)} \alpha\ P[\, |(x_n)_\alpha - c| \geq \epsilon]$$

|||ly $\bigcup_{\alpha\epsilon(0,1)} \alpha\ P((x_n)_\alpha \leq x - c\ \in)$

$$\leq \bigcup_{\alpha\epsilon(0,1)} \alpha\, P((x_n)_\alpha + (y_n)_\alpha \leq x) + \bigcup_{\alpha\epsilon(0,1)} \alpha\, P(|(y_n)_\alpha - c|\ \geq \in)$$

and so

$F(x - c\ \in) \leq \underline{\lim} F_{x_n+y_n}(x)$ (4.2)

since$(x - c)$ is a continuity point of F and $\in > 0$ from (1) and (2)

$$\lim_{n\to\infty} F_{x_n+y_n}(x) = F(x - c)$$

(ii) Let Cx, $(C+\in)x$, $(C-\in)x$ be all continuity points of F.

Then

$$P\left(\frac{x_n}{y_n} \le x\right) = P\left[\bigcup_{\alpha\epsilon(0,1)} \alpha \frac{(x_n)_\alpha}{(y_n)_\alpha} \le x\right]$$
$$= \bigcup_{\alpha\epsilon(0,1)} \alpha\left[\frac{(x_n)_\alpha}{(y_n)_\alpha} \le x\right]$$
$$\le \bigcup_{\alpha\epsilon(0,1)} \alpha\left[P\frac{(x_n)_\alpha}{(y_n)_\alpha} \le x, |(y_n)_\alpha| <\in\right]$$
$$+ \bigcup_{\alpha\epsilon(0,1)} \alpha\,[P|(x_n)_n - c| \ge\in]$$
$$\le \bigcup_{\alpha\epsilon(0,1)} \alpha\,[P(x_n)_\alpha \le (c+\in)x + P(|(y_n)_\alpha - c| \ge\in]$$

Therefore

$$\bigcup_{\alpha\epsilon(0,1)} \alpha\, P\left[\frac{(x_n)_\alpha}{(y_n)_\alpha} \le x\right] \le \bigcup_{\alpha\epsilon(0,1)} \alpha\, P[(x_n)_\alpha \le (C+\in)x]$$
$$\overline{\lim}\, P\left(\frac{x_n}{y_n} \le c\right) \le \overline{\lim}\, P(x_n \le (C+\in)x)$$
$$+ \lim \mathrm{P}\,(|(\mathrm{y_n})_\alpha - \mathrm{c}| \ge\in)$$
$$= F\big((C+\in)x\big) \qquad (4.3)$$

Similarly

$$P(x_n \le (c-\in)x) \le P\left(\frac{x_n}{y_n} \le x\right)$$
$$+ P(|y_n - c| \ge\in)$$

Therefore

$$\underline{\lim} P[x_n \le (c-\in)x] = F((c-\in)x)$$
$$\le \underline{\lim}\, P\left(\frac{x_n}{y_n} \le x\right) \qquad (4.4)$$

Equations (3) and (4) imply (ii)

(iii) Let $\left(\frac{x}{c}\right)\frac{x}{c-\in}, \frac{x}{c+\in}$ be continuity points of F.

Then $F_{x_n+y_n}(x) = P(x_n y_n \le x)$

$$= P\left(\bigcup_{\alpha\in(0,1)} \alpha(x_n)_\alpha (y_n)_\alpha \le x\right)$$
$$= \bigcup_{\alpha\in(0,1)} \alpha\ P((x_n)_\alpha (y_n) \le x)$$
$$\le \bigcup_{\alpha\in(0,1)} \alpha\ P((x_n)_\alpha (y_n)_\alpha \le x),$$
$$|(y_n)_\alpha - c| \le \in$$
$$+ \bigcup_{\alpha\in(0,1)} \alpha\ P\left((x_n)_\alpha \le \frac{x}{c-\in}\right),$$
$$\overline{\lim}\, P(x_n y_n \le x) \le \overline{\lim}\, P\left(x_n \le \frac{x}{c-\in}\right)$$
$$+ \overline{\lim}\, P(|y_n - c| \ge\in)$$
$$= F\left(\frac{x}{c-\in}\right)$$

Similarly

$$P(x_n y_n \le x) + P(|y_n - c| \ge\in)$$
$$\ge P\left(x_n \le \frac{x}{c-\in}\right) \ (4.5)$$

This shows that

$$\underline{\lim}\, P(x_n y_n \le x) + \lim\ P(|y_n - c| \ge\in)$$
$$\ge \underline{\lim} P\left(x_n \le \frac{x}{c+\in}\right)$$

This shows thatx

$$\lim P(x_n y_n \le x) \ge F\left(\frac{x}{c+\in}\right)$$

- Theorem 4.2

  Let $(x_n)$and (y_n) be sequence of fuzzy random variables then

  i. $x_n \overset{P}{\rightarrow} x \implies x_n \overset{d}{\rightarrow} x$

  ii. If X is a constant a.s. then $x_n \overset{d}{\rightarrow} c \implies x_n \overset{P}{\rightarrow} c$.

  iii. Let $P_j(n) = P(x_n = j), P_j = P(x = j)$ where x_n and are integer valued r.r.s. then.

  $P_j(n) \longrightarrow P_j$ for all $j \geq 1$ if and only if $x_n \overset{d}{\rightarrow} x$.

Proof:

i. First we prove that $x_n \overset{P}{\rightarrow} x \implies x_n \overset{d}{\rightarrow} x$

Now $x_n \overset{P}{\rightarrow} x \implies P(|x_n - x| \geq \in) \rightarrow 0$ as $n \rightarrow$ and for all $\in > 0$. Let $x \in c(F)$ where F is the distribution function of X and $\in > 0$ be arbitrary.

Then $P(x_n \leq x) = P(\cup_{\alpha\in(0,1)} \alpha\,(x_n)_\alpha \leq x)$

$$- P\left(\bigcup_{\alpha\in(0,1)} \alpha(x_n)_\alpha \leq x + \in\right)$$

$$= \bigcup_{\alpha\in(0,1)} \alpha\ P(x_n)_\alpha \leq x$$

$$\bigcup_{\alpha\in(0,1)} \alpha\ P((x_n)_\alpha \leq x + \in)$$

$$\leq \bigcup_{\alpha\in(0,1)} \alpha\ P((x_n)_\alpha \leq x, (x_n)_\alpha > x + \in)$$

$$\leq \bigcup_{\alpha\in(0,1)} \alpha\ P((x)_n > (x_n)_\alpha + \in)$$

$$\leq \bigcup_{\alpha\in(0,1)} \alpha\ P[|(x_n)_\alpha - (x)_\alpha| \geq \in]$$

$\rightarrow 0$as$n \rightarrow \infty$ for all $\in > 0$.

$\implies \overline{\lim} F_n(x) - F(x + \in) \leq 0$

$\implies \overline{\lim} F_n(x) \leq F(x + \in)$

Again

$$\bigcup_{\alpha\in(0,1)} \alpha\ P((x_n)_\alpha \leq x - \in) - \bigcup_{\alpha\in(0,1)} \alpha\ P((x_n)_\alpha \leq x)$$

$$\leq \bigcup_{\alpha\in(0,1)} \alpha\ P((x_n)_\alpha \geq x, (x)_\alpha \leq x - \in$$

$$< \bigcup_{\alpha\in(0,1)} \alpha\ P(|(x_n)_\alpha - X| \geq \in) \rightarrow 0$$

as$n \rightarrow \infty$ and for all $\in > 0$.

Hence $\overline{\lim} F_n(x) \geq F(x - \in)$

Now $x \in c(F) = \overset{\lim}{\leftrightarrow}_0\ F(x + \in) = F(x)$

Therefore

$$\overline{\lim} F_n(x) \leq F(x) \leq \underline{\lim} F_n(x)$$

But $\underline{\lim} F_n(x) \leq \overline{\lim} F_n(x)$

Hence $\lim F_n(x) = F(x)$ if $x \in c(F)$

and$x_n \overset{d}{\rightarrow} x$ is proved.

ii. since$x$ is a internal constant c almost surely we can without loss of generality write $x(\omega) = c$ for all $\omega's$.

$$F(x) = \begin{cases} 0 \text{ if } x < c \\ 1 \text{ if } x \geq c \end{cases}$$

so $c$is the internal of discontinuity points of $F(x)$.

$P(\cup_{\alpha\in(0,1)} \alpha\ |(x_n)_\omega - (x)_\alpha| \geq \in)$

$= \cup_{\alpha\in(0,1)} \alpha\ P(|(x_n)_\alpha - (x)_\alpha| \geq \in)$

$= \bigcup_{\alpha\in(0,1)} \alpha\ P(|(x_n)_\alpha - c| \geq \in)$
$= \bigcup_{\alpha\in(0,1)} \alpha\ P((x_n)_\alpha \geq c+\in) + \bigcup_{\alpha\in(0,1)} \alpha\ P((x_n)_\alpha \leq c-\in)$
$= 1 - F_n(c+\in) + F_n(c-\in)$ since $c \pm \in \in c(F)$
$\Longrightarrow 1 + F(C+\in) + F(C-\in) = 0$
$\because F(c+\in) = 1$ and $F(c-\in) = 0$
Therefore $x_n \xrightarrow{P} c$.

- Theorem 4.3

Let $x_1\, x_2 \cdots$ be a sequence of x-dimensional fuzzy random variables and $g: R^k \to R^m$ is a measurable mapping which is continuous over a Borel set $B \subset R^K$ for which $P(x \in B) = 1$ and if $x_n \xrightarrow{P} x$ then $g(x_n) \xrightarrow{P} g(x)$.

→ Proof:
Let $\{g(x_{nj})\}$ be any subsequence of $g(x)$.
we are only to prove that there is a subsequence $g\{x_{rj}\}$ of $g\{x_{nj}\}$ which converges a.s to $g(x)$.
since $x_n \xrightarrow{P} x$ there is a subsequence $g\{x_{rj}\}$ of $g\{x_{nj}\}$ such that $x_{rj} \to x$ a.s.

Let A = $[x_{rj} \to x] \cap [x \in B]$
Then P(A) = 1 and W∈ A implies that

$$X_{rj}(W) \to X(W) \in B$$

Since g is continuous on B than
$g(x_{rj}) \to g(X(w))$ as $j \to \infty$ for all $W \in A$
$g(x_{rj}) \to g(x)\ a.s$
$\therefore g(x_n) \xrightarrow{P} g(x)$

- Theorem 4.4:

Let $x_n \xrightarrow{P} x$ and $y_n \xrightarrow{P} x$ then
i) $x_n + y_n \xrightarrow{P} x + y$ and
ii) $x_n y_n \xrightarrow{P} xy$

→ Proof:
i) $P(|x_n + y_n - (x+y)\} \geq \in)$

$$= P\left(\bigcup_{\alpha\in(0,1)} \alpha|(x_n)_\alpha(y_n)_\alpha - (x_\alpha + y_\alpha)| \geq \in\right)$$
$$= \bigcup_{\alpha\in(0,1)} \alpha\ P(|(x_n)_\alpha(y_n)_\alpha - (x_\alpha + y_\alpha)| \geq \in)$$
$$\leq \bigcup_{\alpha\in(0,1)} \alpha\ P(|(x_n)_\alpha - (x_\alpha)| \geq {}^{\in}/_{2}$$
$$+ \bigcup_{\alpha\in(0,1)} \alpha\ P(|(y_n)_\alpha - (y_\alpha)| \geq {}^{\in}/_{2}$$
$$\Longrightarrow P(|x_n + y_n - (x+y)| \geq \in) \to 0.$$

ii) First assume that x = 0 a.s.
Let $\in > 0$. choose K_o such that $P(|y| \geq K_o) < \eta < 0$ since y is a fuzzy random variable if is finite as

$$[|x_n y_n| \geq \in] = \left[\bigcup_{\alpha\in(0,1)} |(x_n)_\alpha(y_n)_\alpha \geq \in|\right]$$
$$\subset \left[\bigcup_{\alpha\in(0,1)} |x_n| \geq \frac{\in}{K_o + 1}\right]$$

$$\cup \left[\bigcup_{\alpha \in (0,1)} |y_n| \geq K_{0+1} \right]$$

$\Rightarrow \overline{\lim} P(|x_n y_n| \geq \in) \leq \overline{\lim} P(|y_n| \geq K_{0+1})$

$$\overline{\lim} P(|x_n| \geq {}^{\in}/_{K_{0+1}}$$
$$= \overline{\lim} P(|y_n| \geq K_{0+1})$$

Now

$$P\left(\bigcup_{\alpha \in (0,1)} \alpha |(y_n)_\alpha| \geq K_{0+1} \right) = \bigcup_{\alpha \in (0,1)} \alpha P[|(y_n)_\alpha| \geq K_{0+1}]$$
$$\leq \bigcup_{\alpha \in (0,1)} \alpha\ P[|(y_n)_\alpha - y_\alpha| \geq 1]$$
$$+ \cup_{\alpha \in (0,1)} \alpha\ P(|(y)_\alpha| \geq K_0)$$

Therefore
$\overline{\lim} P(|x_n y_n| \geq \in) \rightarrow 0 \text{as} n \rightarrow \infty$
and $\overline{\lim} > \underline{\lim}$
So that $\underline{\lim} P(|x_n y_n| \geq \in) \rightarrow 0$ as $n \rightarrow \infty$
Hence $\lim_{n \to \infty} P(|x_n y_n| \geq \in) = 0$ for all $\in > 0$.
Therefore
$x_n y_n \xrightarrow{P} 0$
In general
$x_n y_n - xy = (x_n - x)(y_n - y) + x(y_n - y) + y(x_n - x)$
and all the terms converge in probability to zero.
So $x_n y_n \xrightarrow{P} xy$

5. References

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