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# Unreliable $M^x/G/1$ Queueing System with Two Types of Repair

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***Abstract:***
*This paper deals with a Single Server non-Markovian batch arrival queueing system ($M^X/G/1$) in which the breakdowns occur during busy period, according to a Poison process and the server is sent for repair immediately. The breakdown server is facilitated with two types of repair, according as the customer just being served stays in the service facility (with probability 1-q) to complete the remaining service or joins the head of the queue and opts for a new service (with probability q). Immediately after the server is fixed, the customer waiting for the completion of the remaining service is considered for service if exists, otherwise the customer in the head of the queue is taken for service. Further, it is assumed that the server is as good as new after repair. The expressions for the steady state system size probabilities when the system is in different states and the corresponding expected system size are derived in a closed form.. The results of some particular cases of interest are discussed in order to verify the results. Moreover, some numerical examples are also presented.*

***Keywords:*** *Repair, supplementary variable technique, unreliable and $M^X/G/1$*

## 1. Introduction

The server failures which lead to, service interruptions are quite common in many real life situations. It is well known that performance measures of unreliable queuing systems are heavily influenced by server failures. For this reason, unreliable queuing systems have been investigated extensively over the decades. The early works with interruptions are due to Thirurengadan (1963) [14], Mitrany and Avi-Itzhak (1968) [13], and references therein. Recently, Choudhury and Ke (2012)[2], Choudhury and Tadj (2011) [3], Dimitriou (2013) [4], Dimitriou and Langaris (2010) [5], Falin (2010) [6], Ke (2004,2005,2006, 2006a)[7,8,9,10], Ke et al. (2009) [11], Lee et al. (2011) [12], Yang et al. (2002)[15], and others considered the unreliable queueing systems with various features wherein one of the underlying assumptions is that a failed server is sent for repair at the repair shop and present customers in the system should wait for the server to be repaired without being served.In most of the queueing models with server breakdowns, the server sent for repair will be provided with only one type of repair facility. But there are situations the breakdown server will be provided with different types of repair facility according to the behavior of the interrupted customers. In the present work the authors consider a $M^X/G/1$ queueing model in which, if the server fails the customer in service may stay in the service facility with probability (1-q) to continue the remaining service or may join the head of the queue and opts for a new service with probability (q) after repair. Accordingly the repair times of the service follow two different heterogeneous distributions. The steady state behavior of the model is analysed using supplementary variable technique. Because of the complexity only very few article are available with two types of repair.

## 2. Mathematical Analysis of the System

### *2.1. Model Description*

#### 2.1.1. Idle Period

A cycle starts, whenever the system empties and the server is turned off and stays idle in the system. The idle period ends as soon as a batch of customers arrives for service. The time during which the server stays idle in the system is called idle period.

2.1.2. Arrivalpattern

Customers arrive in batches in accordance with the time homogeneous Poisson process with group arrival rate $\lambda$. The batch size **X**is a random variable with probability distribution $\Pr(X = k) = g_k$ , k=1, 2, 3…i.e) The probability that a batch of k units arrive in an infinitesimal interval (t, t+h) is $\lambda g_k + O(h).$

2.1.3. Busy Period

Busy period starts at the end of each idle period. The customers are served, one at a time according to the order of their arrivals. It is assumed that the service times follow general distribution S(x), density function s(x) of finite moments $E(S^k)$, k=1, 2 and Laplace Stieltjes Transform $S^*(\theta)$.

2.1.4. Breakdowns and Repair Period

The server is subject to breakdown at any time while serving customers. It is assumed that the life time of the server follows exponential distribution with rate$\boldsymbol{\alpha}$. The breakdown server is immediately sent for repair to a repair facility. Whenever, the server fails the customer in service, either stays in the service facility for the server to return from repair facility to complete the remaining service with probability (1-q) or joins the head of the queue to repeat the service, with probability q. The corresponding repair times of the server respectively follow heterogeneous general distributions $R_1(y)$ and $R_2(y)$ with density functions $r_1(y)$, $r_2(y)$ of finite moments E $(R_i^k)$,i,k=1,2.The server returning from repair facility is considered as good as new. The server continues this type of services until the system becomes empty. i.e) the server is turned off only when the system becomes empty again. The busy period and breakdown period constitute completion period. The system will be turned on again when a new arrival occurs. The completion period and idle period will constitute a cycle.

The arriving customers always join the system and form a single waiting line based on the order of the batches. It is further assumed that the customers within a batch are pre-ordered for service. The customers are served one by one according to the order in the queue. The system is denoted by $M^X$/G/1/BREAKDOWN/TWO TYPES OF REPAIR.

The steady-state system size equations under the steady-state condition are analysed by using supplementary variable technique. The PGFof the system size is obtained in a closed form so that various performance measures can be derived from it.

→ Notation

The following notations are used to discuss the model

N (t)= The system size at time t

$\lambda$ = Group arrival rate

X= Groupsize random variable

$\Pr(X = k) = g_k$ , k=1, 2, 3…

X(z)= Probability generating function of X.

The notations of random Variables (RV), Cumulative Distribution Function (CDF), Probability Density Function (PDF), Laplace-Stieltjes Transform (LST) and its $k^{th}$ momentsused to model the queueing system are listed below

| | RV | CDF | PDF | LST | $k^{th}$ moment |
|---|---|---|---|---|---|
| Service time | S | S(x) | s(x) | $S^*(\theta)$ | $E(S^k)$ |
| Repair time | $R_1$ | $R_1(x)$ | $r_1(x)$ | $R_1^*(\theta_1)$ | $E(R_1^k)$ |
| Repair time | $R_2$ | $R_2(x)$ | $r_2(x)$ | $R_2^*(\theta)$ | $E(R_2^K)$ |

*Table 1*

Where $F^*(\theta) = \int_0^\infty e^{-\theta x} f(x)dx = \int_0^\infty e^{-\theta x} d(F(x))$

Let $S^0(t)$ , $R_1^0(t)$ and $R_2^0(t)$ denote the remaining service time and remaining repair time at time t. Further the states of the system are denoted by the RV Y (t) at time t.

i.e., Y (t) = 0,1,2,3 respectively denotes, the server is in idle, busy, repair mode with and without customer in service facility (remaining service time, remaining repair time). The supplementary variables are introduced in order to obtain a bivariate Markov Process $\{N(t), \delta(t)\}$ where N (t) denotes the system size random variable and $\delta(t) = \{0, S^0(t), R_1^0(t), R_2^0(t)\}$according asY(t) = 0,1,2,3 respectively

Let $R_n(t) = \Pr\{N(t) = n, Y(t) = 0\}$

$P_n(x,t)\,dt = \Pr\{N(t) = n,\ x \le S^0(t) \le x + dt,\ Y(t) = 1\},\ n \ge 1$

$B_{n,1}(x,y,t)\,dt = \Pr\{N(t) = n,\ S^0(t) = x,\qquad y \le R_1^0(t) \le y + dt,\ Y(t) = 2\},\ n \ge 1$

$B_{n,2}(x,t)\,dt = \Pr\{N(t) = n,\ x \le R_2^0(t) \le x + dt,\ Y(t) = 3\},\ n \ge 1$

Then, $R_0(t)$ denote the probability that the system is empty at time t.

$P_n(x,t)$ denotes the probability that there are n-customers in the system at time t, the server is busy and the remaining service time x lies in the interval $[x, x+\Delta t]$

$B_{n,1}(x,y,t)$ denotes the probability that there are n-customers in the system at time t, the server is under repair, the remaining repair time y lies in the interval $[y, y+\Delta t]$ and the customer whose service is terminated due to breakdown has to complete the remaining service time x.

$B_{n,2}(x,t)$ denote the probability that there are n-customers in the system at time t, the server is under repair and the remaining repair time x lies in the interval $[x, x+\Delta t]$ and the customer whose service is terminated due to breakdown joins the head of queue to repeat the service.

Further $P_n(0)$ , $B_{n,1}(0,0)$, $B_{n,2}(0)$ denote the probability that there are n customers in the system at the termination of service time and repair times respectively.

Assuming that at steady-state, probabilities are independent of time t, we have

$$\lim_{t\to\infty} P_n(x,t) = P_n(x),\ \lim_{t\to\infty} B_{n,1}(x,y,t) = B_{n,1}(x,y),\ \lim_{t\to\infty} B_{n,2}(x,t) = B_{n,2}(x),$$

$$\lim_{t\to\infty} \frac{\partial}{\partial x} P_n(x,t) = \frac{d}{dx} P_n(x),$$

$$\lim_{t\to\infty} \frac{\partial}{\partial y} B_{n,1}(x,y,t) = \frac{d}{dx} B_{n,1}(x,y),\ \lim_{t\to\infty} \frac{\partial}{\partial y} B_{n,2}(x,t) = \frac{d}{dx} B_{n,2}(x),$$

$$\lim_{t\to\infty} R_n(t) = R_n,$$

$$\lim_{t\to\infty} \left(\frac{\partial}{\partial t} P_n(x,t) = \frac{\partial}{\partial t} B_{n,1}(x,y,t) = \frac{\partial}{\partial t} B_{n,2}(x,t)\right) = 0$$

*2.2. The System Size Distribution*

The following steady-state equations are obtained for queueing system, using supplementary variable technique, and following the argument of Cox (1955)[1],

- Idle state

$$\lambda R_0 = P_1(0)$$

- Busy state

$$-\frac{d}{dx} P_n(x) = -(\lambda+\alpha) P_n(x) + P_{n+1}(0)\, s(x) + \lambda \sum_{k=1}^{n-1} P_{n-k}(x)\, g_k + \lambda R_0 g_n\, s(x) + B_{n,1}(x,0) + B_{n,2}(0)\, s(x),\ n \ge 1$$

- Breakdown state

$$-\frac{d}{dy} B_{n,1}(x,y) = -\lambda B_{n,1}(x,y) + \lambda \sum_{k=1}^{n-1} B_{n-k,1}(x,y)\, g_k$$

$$+\alpha(1-q) P_n(x) r_1(y),\ n \ge 1$$

$$-\frac{d}{dx} B_{n,2}(x) = -\lambda B_{n,2}(x) + \lambda \sum_{k=1}^{n-1} B_{n-k,2}(x)\, g_k,$$

$$+\alpha q \int_0^\infty P_n(w)\, dw\, r_2(x),\ n \ge 1$$

The L.S.T of the steady-state equations are obtained by using the definition of Laplace- Stieltjes Transformation and its properties. The L.S.T of the density functions are defined in the previous table. The remaining notations of the L.S.T are listed below

| Probability Distribution | L.S.T |
|---|---|
| $P_n(x)$ | $P_n^*(\theta)=\int_0^\infty e^{-\theta x}P_n(x)\,dx$ |
| $B_{n,1}(x,y)$ | $B_{n,1}^*(\theta,y)$ |
| $B_{n,1}^*(\theta,y)$ | $B_{n,1}^{**_1}(0,\theta_1)=\int_0^\infty e^{-\theta_1 y}B_{n,1}^{**_1}(\theta,y)\,dy$ |
| $B_{n,2}(x)$ | $B_{n,2}^*(\theta)$ |

*Table 2*

Thus the L.S.T of the equations with respect to x are given by,

$$\theta P_n^*(\theta)-P_n(0)=(\lambda+\alpha)P_n^*(\theta)-P_{n+1}(0)S^*(\theta)$$
$$-\lambda\sum_{k=1}^{n-1}P_{n-k}^*(\theta)g_k-\lambda R_0 g_n S^*(\theta)-B_{n,1}^*(\theta,0)\ -B_{n,2}(0)S^*(\theta),\ n\geq 1\quad (1)$$
$$-\frac{d}{dy}B_{n,1}^*(\theta,y)=-\lambda B_{n,1}^*(\theta,y)+\lambda\sum_{k=1}^{n-1}B_{n-k,1}^*(\theta,y)g_k+\alpha(1-q)P_n^*(\theta)r_1(y),\ n\geq 1$$
(2)
$$\theta B_{n,2}^*(\theta)-B_{n,2}(0)=\lambda B_{n,2}^*(\theta)-\lambda\sum_{k=1}^{n-1}B_{n-k,2}^*(\theta)g_k\quad -\alpha q\int_0^\infty P_n(w)R_2^*(\theta),\ n\geq 1\quad (3)$$

And the L.S.T of the equation (2) with respect to y are given by,

$$\theta_1 B_{n,1}^{**_1}(\theta,\theta_1)-B_{n,1}^*(\theta,0)=\lambda B_{n,1}^{**_1}(\theta,\theta_1)$$
$$-\lambda\sum_{k=1}^{n-1}B_{n-k,1}^{**_1}(\theta,\theta_1)g_k-\alpha(1-q)P_n^*(\theta)R_1^{*_1}(\theta_1)\quad n\geq 1\ (4)$$
$$\theta B_{n,2}^*(\theta)-B_{n,2}(0)=\lambda B_{n,2}^*(\theta)-\lambda\sum_{k=1}^{n-1}B_{n-k,2}^*(\theta)g_k\quad -\alpha q\int_0^\infty P_n(w)R_2^*(\theta),\ n\geq 1$$

*2.3. Probability Generating Functions*

Now to obtain the partial PGF's of the number of customers in the system, the following partial PGF's are defined

$$P^*(z,\theta)=\sum_{n=1}^\infty P_n^*(\theta)z^n,\ P(z,0)=\sum_{n=1}^\infty P_n(0)z^n$$
$$B_1^{**_1}(z,\theta,\theta_1)=\sum_{n=1}^\infty B_{n,1}^{**_1}(\theta,\theta_1)z^n\quad B_1^*(z,\theta,0)=\sum_{n=1}^\infty B_{n,1}^*(\theta,0)z^n$$
$$B_2^*(z,\theta)=\sum_{n=1}^\infty B_{n,2}^*(\theta)z^n,\ B_2(z,0)=\sum_{n=1}^\infty B_{n,2}(0)z^n$$

The partial PGFS are obtained by multiplying the corresponding equations with suitable powers of z and following some algebraic manipulations.

The identity $\sum_{n=2}^\infty z^n\sum_{k=1}^{n-1}a_{n-k}b_k=\left(\sum_{n=1}^\infty a_n z^n\right)\left(\sum_{n=1}^\infty b_n z^n\right)$ is used to derive the PGF's.

Multiplying the equation (3) by suitable powers of z and adding $n=0$ to $\infty$ we get,

$$\theta\sum_{n=1}^\infty B_{n,2}^*(\theta)z^n-\sum_{n=1}^\infty B_{n,2}(0)z^n=\lambda\sum_{n=1}^\infty B_{n,2}^*(\theta)z^n$$
$$-\lambda\sum_{n=2}^\infty z^n\sum_{k=1}^{n-1}B_{n-k,2}^*(\theta)g_k-\alpha\, qR_2^*(\theta)\sum_{n=1}^\infty P_n^*(0)z^n$$
$$\theta B_2^*(z,\theta)-B_2(z,0)=\lambda B_2^*(z,\theta)-\lambda B_2^*(z,\theta)X(z)$$
$$-\alpha\, qP^*(z,0)R_2^*(\theta)$$
$$B_2^*(z,\theta)[\theta-\lambda(1-X(z))]=B_2(z,0)-\alpha\, q\, P^*(z,0)R_2^*(\theta)\quad B_2^*(z,\theta)[\theta-w_X(z)]=B_2(z,0)-\alpha\, q\, P^*(z,0)R_2^*(\theta)$$
(5) where $w_X(z)=\lambda(1-X(z))$ (5.1)

At $\theta=w_X(z)$ we get,

$$B_2(z,0)=\alpha\, qP^*(z,0)R_2^*(w_X(z))\quad (6)$$

Substituting the value of $B_2(z,0)$ in equation (5),

$$B_2^*(z,\theta) = \frac{\alpha qP^*(z,0)(R_2^*(w_X(z)) - R_2^*(\theta))}{\theta - w_X(z)} \quad (7)$$

At $\theta = 0$, $R^*(\theta) = 1$ we get

$$B_2^*(z,0) = \frac{\alpha qP^*(z,0)(1 - R_2^*(w_X(z)))}{w_X(z)} \quad (8)$$

Next to determine the PGF $B_1^{**_1}(z,\theta,\theta_1)$, equation (4) is used,

Multiplying the equation (4) by$z^n$ and adding $n = 1 \, to \, \infty$ we get,

$$\theta_1 B_1^{**_1}(z,\theta,\theta_1) - B_1^*(z,\theta,0) = \lambda B_1^{**_1}(z,\theta,\theta_1)$$
$$- \lambda B_1^{**_1}(z,\theta,\theta_1)X(z) - \alpha(1-q)P^*(z,\theta)R_1^{*_1}(\theta_1)$$
$$B_1^{**_1}(z,\theta,\theta_1)[\theta_1 - \lambda(1 - X(z))] = B_1^*(z,\theta,0)$$
$$- \alpha(1-q)P^*(z,\theta)R_1^{*_1}(\theta_1)$$
$$B_1^{**_1}(z,\theta,\theta_1)[\theta_1 - w_X(z)] = B_1^*(z,\theta,0)$$
$$- \alpha(1-q)P^*(z,\theta)R_1^{*_1}(\theta_1) \quad (9)$$

At $\theta_1 = w_X(z)$ in equation (9), we get

$$B_1^*(z,\theta,0) = \alpha(1-q)P^*(z,\theta)R_1^{*_1}(w_X(z)) \quad (10)$$

Substituting the value of $B_1^*(z,\theta,0)$ in equation (9) we get,

$$B_1^{**_1}(z,\theta,\theta_1) = \frac{\alpha(1-q)P^*(z,\theta)(R_1^{*_1}(w_X(z)) - R_1^{*_1}(\theta_1))}{\theta_1 - w_X(z)} \quad (11)$$

At $\theta_1 = 0$, $R^{*_1}(\theta) = 1$ and $\theta = 0$,

$$B_1^{**_1}(z,0,0) = \frac{\alpha(1-q)P^*(z,0)(1 - R_1^{*_1}(w_X(z)))}{w_X(z)} \quad (12)$$

Similarly multiplying the equation (1) by suitable powers of z and adding over $n = 1 to \infty$ the PGF of the system size when the server is busy is obtained.

i.e., $\theta P^*(z,\theta) - P(z,0) = (\lambda + \alpha)P^*(z,\theta) - \lambda X(z)P^*(z,\theta) \quad - \frac{1}{z}[P(z,0) - P_1(0)z]S^*(\theta) - \lambda R_0 X(z)S^*(\theta)$

$$- B_1^*(z,\theta,0) - B_2(z,0)S^*(\theta)$$

Substituting for $P_1(0) = \lambda R_0$, for $B_1^{**_1}(z,0,0)$ and $B_2^*(z,0)$ from equations (10) and (8) respectively, we get

$$P^*(z,\theta)[\theta - \lambda(1 - X(z)) + \alpha] = P(z,0)\left[\frac{z - S^*(\theta)}{z}\right]$$
$$- \alpha qP^*(z,0)R_2^{*_1}(w_X(z))S^*(\theta) + \lambda R_0(1 - X(z))S^*(\theta) - \alpha(1-q)P^*(z,\theta)R_1^{*_1}(w_X(z))S^*(\theta)$$
$$P^*(z,\theta)[\theta - (w_X(z) + \alpha(1 - (1-q)R_1^{*_1}(w_X(z))))]$$
$$= P(z,0)\left[\frac{z - S^*(\theta)}{z}\right] + R_0 w_X(z)S^*(\theta)$$
$$- \alpha qP^*(z,\theta)R_2^*(w_X(z))S^*(\theta)$$
$$P^*(z,\theta)[\theta - h_\alpha(w_X(z))] = P(z,0)\left[\frac{z - S^*(\theta)}{z}\right] + R_0 w_X(z)S^*(\theta) - \alpha qP^*(z,\theta)R_2^*(w_X(z))S^*(\theta) \quad (13)$$

Where $h_\alpha(w_X(z)) = w_X(z) + \alpha(1 - (1-q)R_1^{*_1}(w_X(z)))$ (13.1)

At $\theta = 0$ the equation (13) leads to,

$$P^*(z,0) = \frac{\frac{P(z,0)}{z}(z-1) + R_0 w_X(z)}{\alpha qR_2^{*_1}(w_X(z)) - h_\alpha(w_X(z))} \quad (14)$$

Substituting (14) in (13) we get,

$$P^*(z,\theta)[\theta - h_\alpha(w_X(z))] = \frac{P(z,0)}{z}[z\alpha qR_2^*(w_X(z))$$
$$(1 - S^*(\theta)) + h_\alpha(w_X(z)(S^*(\theta) - z)]$$
$$- R_0 w_X(z)S^*(\theta)h_\alpha(w_X(z) \quad (15)$$

At $\theta = h_\alpha(w_X(z))$ equation (15) gives,

$$\frac{P(z,0)}{z} = \frac{R_0 w_X(z) S^*(h_\alpha(w_X(z)) h_\alpha(w_X(z))}{(S^*(h_\alpha(w_X(z)) - 1)[h_\alpha(w_X(z)) - z\alpha q R_2^*(w_X(z))] + h_\alpha(w_X(z))[1-z]} \quad (16)$$

Substituting (16) in (14) we get

$$P^*(z,0) = \frac{zR_0 w_X(z)(1 - S^*(h_\alpha(w_X(z)))}{(S^*(h_\alpha(w_X(z)) - 1)[h_\alpha(w_X(z)) - z\alpha q R_2^*(w_X(z))] + h_\alpha(w_X(z))[1-z]} \quad (17)$$

Thus the partial PGF's of the system size of the model are listed by

$$P^*(z,0) = \frac{zR_0 w_X(z)(1 - S^*(h_\alpha(w_X(z)))}{(S^*(h_\alpha(w_X(z)) - 1)[h_\alpha(w_X(z)) - z\alpha q R_2^*(w_X(z))] + h_\alpha(w_X(z))[1-z]} \quad (18)$$

$$B_1^{**_1}(z,0,0) = \frac{\alpha(1-q)P^*(z,0)(1 - R_1^{*_1}(w_X(z)))}{w_X(z)} \quad (19)$$

$$B_2^*(z,0) = \frac{\alpha q P^*(z,0)(1 - R_2^*(w_X(z)))}{w_X(z)} \quad (20)$$

To derive the total PGFP(z) of the system size distribution, the following generating functions are considered.

Let $P_{comp}(z)$ = The PGF of the system size when server is busy or in breakdown state

$= P^*(z,0) + B_1^{**_1}(z,\theta,0) + B_2^*(z,0)$

$= \frac{P^*(z,0)}{w_X(z)}[h_\alpha(w_X(z)) - \alpha q R_2^*(w_X(z))]$ (From equations (19) and (20))

$P_I(z) = R_0$ gives the probability that server is idle state

Therefore, the total Probability Generating Function (PGF) of system size distribution (P(z)) at steady-state is given by

$P(z) = P_I(z) + P_{comp}(z)$

$$P(z) = \frac{R_0[S^*(h_\alpha(w_X(z))) h_\alpha(w_X(z)) - zh_\alpha(w_X(z)) S^*(h_\alpha(w_X(z)))]}{(S^*(h_\alpha(w_X(z))) - 1)[h_\alpha(w_X(z)) - z\alpha q R_2^*(w_X(z))] + h_\alpha(w_X(z))[1-z]}$$

Then P(z) can be written as,

$$P(z) = \frac{R_0 h_\alpha(w_X(z)) S^*(h_\alpha(w_X(z)))(z-1)}{D(z)} \quad (21)$$

where $D(z) = h_\alpha(w_X(z))(z - S^*(h_\alpha(w_X(z))))$

$$- z\alpha q R_2^*(w_X(z))(1 - S^*(h_\alpha(w_X(z)))) \quad (21.1)$$

*2.3. Stability Condition*

The total PGF of the system size distribution of the unreliable $M^X/G/1$ queueing system with two types of repair is obtained in terms of the unknown $R_0$ in the equation (21).

$R_0$ can be evaluated using the normalizing condition

$P(1) = 1$ implies, $1 = \lim_{z \to 1} P(z)$

At $z = 1, h_\alpha(w_X(z)) = \alpha q$

Therefore, $P(z) = 1 \Rightarrow 1 = R_0(\alpha q) S^*(\alpha q)\left(\lim_{z \to 1} \frac{z-1}{D(z)}\right)$

i.e., $1 = \frac{R_0 \alpha q S^*(\alpha q)}{D'(1)}$ (Using L' Hospital rule) (22)

Using the results

$$\left(\frac{d}{dz}(h_\alpha(w_X(z))\right)_{z=1} = -\lambda E(X)[1+\alpha(1-q)E(R_1)]$$

$$\left(\frac{d}{dz}(S^*(h_\alpha(w_X(z)))\right)_{z=1}$$

$$= -\lambda E(X)[1+\alpha(1-q)E(R_1)]S^{*'}(\alpha q)$$

( the prime ( ’) symbol denotes the derivative of the function)

$$D'(1) = \frac{d}{dz}D(z)\Big]_{z=1} = \alpha q S^*(\alpha q)(1-\rho) \qquad (23)$$

where $\rho = \lambda E(X)\dfrac{1-S^*(\alpha q)}{\alpha q S^*(\alpha q)}[1+\alpha(1-q)E(R_1)+\alpha q E(R_2)]$

Thus the equation (22) implies

$$R_0 = 1-\rho$$

Therefore the total PGF of thesteady-state system size probabilities of the present model is given by,

$$P(z) = \frac{(1-\rho)h_\alpha(w_X(z))S^*(h_\alpha(w_X(z)))(z-1)}{D(z)} \qquad (24)$$

WhereD(z) is given in equation (21.1)

The equation (24) shows that the probability of the steady-state system size probabilities exist if $\rho<1$ so that $(1-\rho)\succ 1$. Thus $\rho<1$ is the stability condition of the model.

## 3. Performance Measures

In this section, some useful performance measures of the proposed model including the mean queue length are calculated.

### *3.1. The Server is in Busy State*

The probability that the server is busy ( $P_{Busy}$ ) and expected number of customer( $L_{Busy}$ ) in the system when the server is busy are obtained,

$$P_{Busy} = \lim_{z\to 1} P^*(z,0)$$

$$= (1-\rho)(1-S^*(\alpha q))\left(\lim_{z\to 1}\frac{-w_X(z)}{D(z)}\right)$$

(From equation (8))

$$= (1-\rho)(1-S^*(\alpha q))\frac{\lambda E(X)}{D'(1)}$$

Substituting for $D'(1)$ from the equation (23),

$$P_{Busy} = \frac{\lambda E(X)}{\alpha q S^*(\alpha q)}(1-S^*(\alpha q)) \qquad (25)$$

$$L_{Busy} = \left[\frac{d}{dz}P^*(z,0)\right]_{z=1}$$

$$= P_{Busy} + (1-\rho)\left[\lim_{z\to 1}\left(\frac{-w_X(z)}{D(z)}\right)\frac{d}{dz}(1-S^*(h_\alpha(w_X(z)))_{z=1} + \frac{d}{dz}\left(\frac{-w_X(z)}{D(z)}\right)(1-S^*(\alpha q))\right]$$

For the further simplifications we note the following results,

$$\lim_{z\to 1}\left(-\frac{w_X(z)}{D(z)}\right) = \frac{\lambda E(X)}{D'(1)}$$

$$\frac{d}{dz}\left[-\frac{w_X(z)}{D(z)}\right] = \frac{D'(1)\lambda E(X(X-1)) + \lambda E(X)(-D''(1))}{2(D'(1))^2}$$

$$\frac{d}{dz}S^*(h_\alpha(w_X(z))_{z=1} = -\lambda E(X)G_1 S^{*'}(\alpha q)$$

Where $G_1 = [1+\alpha(1-q)E(R_1)]$

Then, $L_{Busy} = P_{Busy} + \dfrac{\lambda E(X(X-1))(1-S^*(\alpha q))}{2\alpha q S^*(\alpha q)}$

$$+\frac{(\lambda E(X))^2[1+\alpha(1-q)E(R_1)]S^{*'}(\alpha q)}{\alpha q S^*(\alpha q)}$$

$$+\frac{(1-S^*(\alpha q))\lambda E(X)(-D''(1))}{2\alpha^2 q^2(1-\rho)} \quad (26)$$

Where $-D''(1)=\lambda E(X(X-1))(1-S^*(\alpha q))[1+\alpha(1-q)E(R_1)$

$$+\alpha qE(R_2)]+(\lambda E(X))^2[(1-S^*(\alpha q))[\alpha(1-q)E(R_1{}^2)+\alpha qE(R_2{}^2)]+2S^{*'}(\alpha q)[1+\alpha(1-q)E(R_1)]$$
$$[1+\alpha(1-q)E(R_1)+\alpha qE(R_2)]]+2\lambda E(X)[(1-S^*(\alpha q))\alpha qE(R_2)$$
$$+(1+\alpha(1-q)E(R_1))(1+\alpha qS^{*'}(\alpha q))] \quad (26.1)$$

*3.2. The Server in Breakdown State*

The probability that the server is in breakdown state of type i( $P_{BRi}$ )i=1, 2 and the expected number of customers in the corresponding state of type i( $L_{BRi}$ ), i=1,2 are obtained by,

$P_{BR1}$ = Probability that the server is in breakdown state and the customer is waiting in the service facility to complete the remaining service time

$$=\lim_{z\to 1} B_1^{**_1}(z,\theta,0)$$

$$=\lim_{z\to 1}\alpha(1-q)P^*(z,0)\frac{(1-R_1^{*_1}(w_X(z)))}{w_X(z)}$$

$$=\alpha(1-q)E(R_1)P_{Busy}$$

Since $\lim_{z\to 1}\frac{1-R_1^{*_1}(w_X(z))}{w_X(z)}=E(R_1)$

Substituting for $P_{Busy}$ from the equation (25) we have,

$$P_{BR1}=\lambda E(X)E(R_1)\frac{(1-q)}{q}\frac{(1-S^*(\alpha q))}{S^*(\alpha q)} \quad (27)$$

$$L_{BR1}=\left[\frac{d}{dz}B_1^{**_1}(z,\theta,0)\right]_{z=1}$$

$$=\alpha(1-q)\left[\lim_{z\to 1}P^*(z,0)\frac{d}{dz}\left(\frac{1-R_1^{*_1}(w_X(z))}{w_X(z)}\right)_{z=1}\right.$$
$$\left.+\lim_{z\to 1}\frac{1-R_1^{*_1}(w_X(z))}{w_X(z)}\frac{d}{dz}(P^*(z,0))_{z=1}\right]$$ Using the result

$$\frac{d}{dz}\left(\frac{1-R_1^{*_1}(w_X(z))}{w_X(z)}\right)_{z=1}=\lambda E(X)\frac{E(R_1^2)}{2}$$

$$L_{BR1}=\alpha(1-q)[\lambda E(X)\frac{E(R_1{}^2)}{2}P_{Busy}+E(R_1)L_{Busy}] \quad (28)$$

$P_{BR2}$ = Probability that the server is in breakdown state and the customer rejoins the head of queue

$$=\lim_{z\to 1}B_2^*(z,0)$$

$$=\alpha q\lim_{z\to 1}P^*(z,0)\frac{(1-R_2^*(w_X(z)))}{w_X(z)}$$

$$=\alpha qE(R_2)P_{Busy}$$

$$P_{BR2}=\lambda E(X)E(R_2)\frac{(1-S^*(\alpha q))}{S^*(\alpha q)} \quad (29)$$

$$L_{BR2}=\left[\frac{d}{dz}B_2^*(z,0)\right]_{z=1}$$

$$= \alpha q \left[ \lim_{z \to 1} P^*(z,0) \frac{d}{dz} \left( \frac{1 - R_2^*(w_X(z))}{w_X(z)} \right)_{z=1} \right.$$

$$\left. + \lim_{z \to 1} \frac{1 - R_2^*(w_X(z))}{w_X(z)} \frac{d}{dz} (P^*(z,0))_{z=1} \right]$$

$$L_{BR2} = \alpha q [\lambda E(X) \frac{E(R_2^2)}{2} P_{Busy} + E(R_2) L_{Busy}] \quad (30)$$

Thus the probability that the system is in breakdown state $P_{BR}$ and the mean queue length $L_{BR}$ are given by

$$P_{BR} = P_{BR1} + P_{BR2}$$

$$L_{BR} = L_{BR1} + L_{BR2}$$

*2.3. Mean System Size*

The average system size (L) of the model is given by

$$L = \frac{d}{dz} [P(z)]_{z=1}$$

$$= (1 - \rho) \left[ \lim_{z \to 1} h_\alpha(w_X(z)) S^*(h_\alpha(w_X(z))) \frac{d}{dz} \left( \frac{z-1}{D(z)} \right)_{z=1} + \lim_{z \to 1} \frac{z-1}{D(z)} \frac{d}{dz} (h_\alpha(w_X(z)) S^*(h_\alpha(w_X(z))))_{z=1} \right]$$

Using the following results

$$\frac{d}{dz} \left( \frac{z-1}{D(z)} \right)_{z=1} = \frac{-D''(1)}{2(D'(1))^2}$$

$$\frac{d}{dz} \Big( h_\alpha(w_X(z)) S^*(h_\alpha(w_X(z))) \Big)_{z=1}$$

$$= (-\lambda E(X))(1 + \alpha_1(1-q)E(R_1)) \frac{S^*(\alpha q) + S^{*'}(\alpha q)}{\alpha q S^*(\alpha q)}$$

$$L = (-\lambda E(X))(1 + \alpha(1-q)E(R_1)) \frac{S^*(\alpha q) + S^{*'}(\alpha q)}{\alpha q S^*(\alpha q)} + \frac{-D''(1)}{2\alpha q(1-\rho)S^*(\alpha q)} \quad (31)$$

- The Expected Waiting Time in System E(W)

$$E(W) = \frac{L}{\lambda E(X)}$$

**4. Particular Cases**

The Steady-state results of M/G/1 and $M^X$/M/1 queueing models with two types of repair are obtained in this section.

*4.1. Markovian Queueing System $M^X$/M/1 with two Types of Repair*

The PGF $(P^R_{M^X/M/1}(z))$ of the corresponding Markovian queueing system with two types of repair and the excepted system size are obtained by putting

$$S^*(h_\alpha(w_X(z))) = \frac{\mu}{\mu + h_\alpha(w_X(z))}$$

$$R^*(w_X(z))) = \frac{\beta}{\beta + \lambda(1 - X(z))}.$$

$$P^R{}_{M^X/M/1}(z) = \frac{(1-\rho)(z-1)\mu(\beta + w_X(z))}{(\beta + w_X(z))[\mu(z-1) + zh_\alpha(w_X(z))] - z\alpha\beta q}$$

*4.2. M/G/1queueing system with two types of repair*

The PGF $P^R{}_{M^X/G/1}(z)$ of the M/G/1 queueing system with two types of repair can be obtained by putting $X(z) = z$ in equation (23) we get

$$P(z) = \frac{(1-\rho)h_{\alpha}(\lambda(1-z))S^{*}(\alpha + (\lambda(1-z)))(z-1)}{h_{\alpha}(w_X(z))(z - S^{*}(\alpha + \lambda(1-z))) - z\alpha q R_2^{*}(\lambda(1-z))(1 - S^{*}(\alpha + (\lambda(1-z))))}$$

**5. Numerical Analysis**

In this section, we study the influence of the system parameters on some significant performance measures through numerical results for the model $M^X$/ G/ 1 with server Breakdown where the unreliable server is provided with two types of repair facilities. The distributions considered for different random variables are listed in the following table.

| Random Variables (Y) | | Distribution F(Y) | Mean E(Y) | Second order moments $E(Y^2)$ |
|---|---|---|---|---|
| Service Time | | Two-stage hyper-exponential | $E(S) = \frac{b}{\mu_1} + \frac{1-b}{\mu_2}$ $0 \le b \le 1$ | $E(S^2) = 2\left(\frac{b}{\mu_1^2} + \frac{1-b}{\mu_2^2}\right)$ |
| Repair time | $R_1$ | Erlang-2 type | $1/\beta_1$ | $3/2\beta_1^2$ |
| | $R_2$ | Deterministic | $1/\beta_2$ | $1/\beta_2^2$ |
| Life time of the server | | Exponential with parameter (a) | $1/a$ | $2/a^2$ |
| Batch size (X) | | Geometric (Geo(p)) | $1/1-p$ | $2p/(i-p)^2$ |

*Table 3*

The influences of $\lambda, q$ and $\alpha$ on

i. System size probabilities when the server is in different states,
ii. The mean queue length (L) and
iii. Expected waiting time of a customer (E(W))

Are analysed through numerical values.

The common parameters for all the tables are $\mu_1 = 15, \mu_2 = 12, \beta_1 = 3, \beta_2 = 2.$

The Table 4 values and Figure 2 show that the mean queue length L and hence expected waiting time E(W) both increase with $\lambda$ and Figure1 represents probabilities for M/G/1 model.

| $\lambda$ | $\rho$ | $P_{Busy}$ | $P_{BR}$ | L | E(W) |
|---|---|---|---|---|---|
| 4 | 0.5612 | 0.3118 | 0.2494 | 1.9555 | 0.4888 |
| 4.5 | 0.6314 | 0.3507 | 0.2806 | 2.6927 | 0.5983 |
| 5 | 0.7015 | 0.3897 | 0.3118 | 3.7969 | 0.7593 |
| 5.5 | 0.7717 | 0.4287 | 0.3429 | 5.6065 | 1.0193 |
| 6 | 0.8419 | 0.4677 | 0.3741 | 9.0606 | 1.5101 |
| 6.5 | 0.9120 | 0.5067 | 0.4053 | 18.0952 | 2.7838 |

*Table 4:* $(p = 0, \alpha = 2, q = 0.4, b = 0.32)$

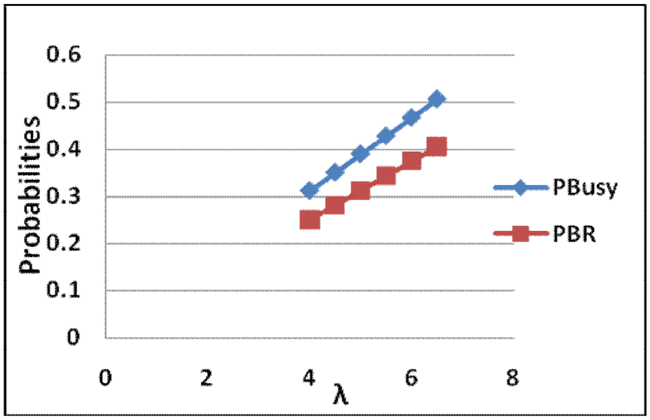


*Figure 1: λ Vs Probabilities*

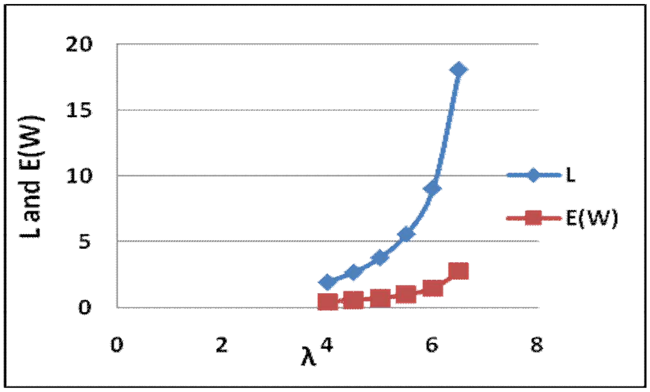


*Figure 2: λ Vs L and E (W)*

In table 5, it is found that the mean system size increases with q where q is the probability with which the customers join the waiting line to repeat the service (from the beginning) soon after the breakdown. This is verified for **$M^X$/M/1** model.The graphical representation is shown in Figures 3 and 4.

| q | $\rho$ | $P_{Busy}$ | $P_{BR}$ | L | E(W) |
|---|---|---|---|---|---|
| 0.25 | 0.7777 | 0.4444 | 0.3333 | 8.9785 | 1.3467 |
| 0.35 | 0.7925 | 0.4444 | 0.3481 | 9.4275 | 1.4141 |
| 0.45 | 0.8074 | 0.4444 | 0.3629 | 10.1918 | 1.5287 |
| 0.55 | 0.8222 | 0.4444 | 0.3777 | 11.2010 | 1.6801 |
| 0.65 | 0.8370 | 0.4444 | 0.3925 | 12.4593 | 1.8689 |
| 0.75 | 0.8518 | 0.4444 | 0.4074 | 14.0095 | 2.1014 |
| 0.85 | 0.8666 | 0.4444 | 0.4222 | 15.9302 | 2.3895 |

*Table 5:* $(p = 0.4, \alpha = 2, \lambda = 4, b = 1)$

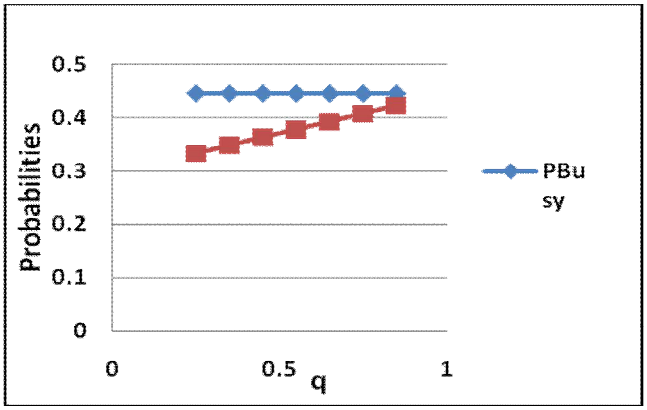


*Figure 3: q Vs Probabilities*

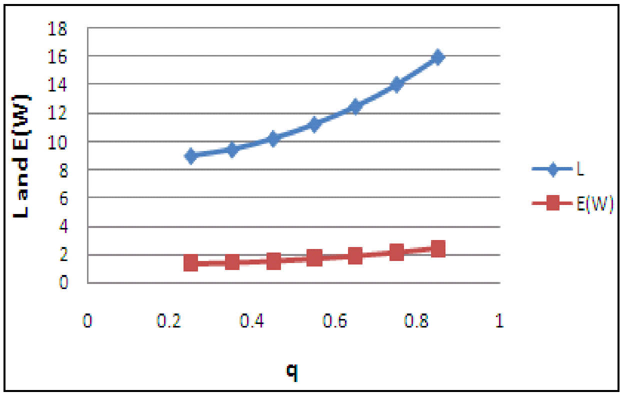


*Figure 4: q Vs L and E(W)*

One can note from Table 6 that the system becomes significantly congested for higher breakdown rate $\alpha$ . Figures 5 and 6 show its graphical representation. The parameters for the Tables 2 and 3 are same as in Table 4 for **$M^X$/G/1** model

| $\alpha$ | $\rho$ | $P_{Busy}$ | $P_{BR}$ | L | E(W) |
|---|---|---|---|---|---|
| 2.5 | 0.5845 | 0.2922 | 0.2922 | 3.0102 | 0.8027 |
| 3 | 0.6429 | 0.2922 | 0.3507 | 3.8579 | 1.0287 |
| 3.5 | 0.7013 | 0.2922 | 0.4090 | 5.0789 | 1.3543 |
| 4 | 0.7596 | 0.2921 | 0.4674 | 6.9293 | 1.8478 |
| 4.5 | 0.8179 | 0.2921 | 0.5258 | 10.0021 | 2.6672 |
| 5 | 0.8763 | 0.2921 | 0.5842 | 16.0151 | 4.2706 |
| 5.5 | 0.9346 | 0.2920 | 0.6425 | 32.8203 | 8.7520 |

*Table 6:* $(p = 0.4, q = 0.4, \lambda = 2.25, a = 0.32)$

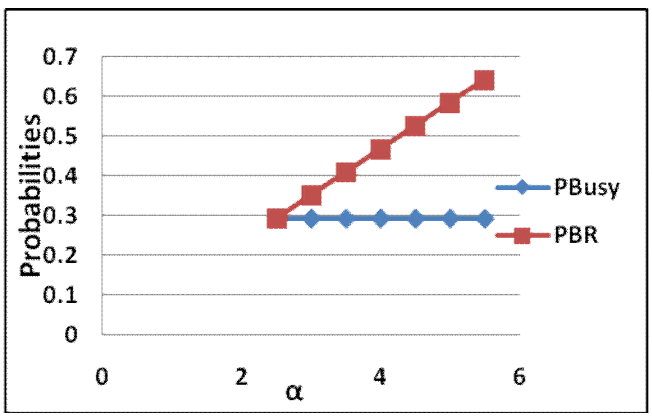


*Figure 5: αVs Probabilities*

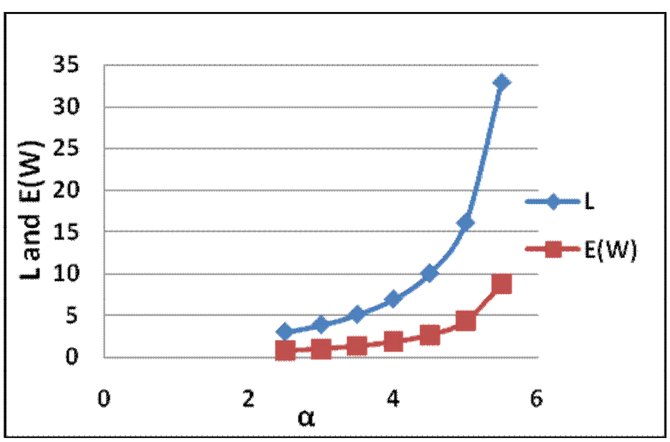


*Figure 6: α Vs L and E(W)*

The similar effects of $\lambda, q$ and $\alpha$ on L for Poisson arrival queueing model (M/G/1) and the Markovian queueing model $M^X/M/1$ are presented in tables 4 and 5.

| $\lambda = 4\ (p = 0, b = 0.32)$ | | | | | |
|---|---|---|---|---|---|
| Q / $\alpha$ | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
| 2 | 2.3756 | 1.9555 | 1.9824 | 2.1343 | 2.3497 |
| 2.5 | 2.7042 | 2.4912 | 2.6967 | 3.0451 | 3.4933 |
| 3 | 3.2589 | 3.3066 | 3.7987 | 3.7987 | 3.7987 |
| 3.5 | 4.1206 | 4.5778 | 5.6271 | 5.6271 | 9.5757 |
| $\lambda = 4.5$ | | | | | |
| q / $\alpha$ | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
| 2 | 3.0815 | 2.6927 | 2.8250 | 3.1211 | 3.5189 |
| 2.5 | 3.7319 | 3.6825 | 4.1667 | 4.9033 | 5.8901 |
| 3 | 4.8665 | 5.3799 | 6.6344 | 8.5743 | 11.641 |
| 3.5 | 6.8718 | 8.6704 | 12.3522 | 20.054 | 44.721 |

*Table 7: Mean queue length with respect to α and q*

| $\lambda = 4\ (p = 0.4, b = 1)$ | | | | | |
|---|---|---|---|---|---|
| Q / $\alpha$ | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
| 1 | 5.4927 | 4.3086 | 4.1324 | 4.2259 | 4.4444 |
| 1.25 | 5.7836 | 4.9821 | 5.0665 | 5.4166 | 5.9179 |
| 1.5 | 6.4233 | 6.0114 | 6.4448 | 7.1972 | 8.1986 |
| 1.75 | 7.4385 | 7.5184 | 8.5066 | 9.9984 | 12.051 |
| $\lambda = 4.5$ | | | | | |
| q / $\alpha$ | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
| 1 | 7.1821 | 6.0155 | 6.0127 | 6.3505 | 6.8750 |
| 1.25 | 8.0407 | 7.4799 | 8.0071 | 8.9583 | 10.254 |
| 1.5 | 9.6336 | 9.8959 | 11.402 | 13.734 | 17.140 |
| 1.75 | 12.323 | 14.095 | 17.982 | 24.721 | 37.932 |

*Table 8: Mean queue length with respect to α and q*

## 6. Conclusion

This paper analyses a $M^X/G/1$ queueing model in which the breakdowns occur according to a Poisson process and the breakdown server is sent to the repair facility. The customer being served may decide either to stay in the service facility to complete the remaining service time after the server is fixed or joins the head of queue to repeat the service from the beginning. Accordingly the repair times are assumed to follow different distributions. The steady state solutions for the model are obtained in a packed form to evaluate the performance measures easily. Numerical computations are provided to study the effect of parameters on system performance measures which validates the analytical results.

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