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# Zero-Free Regions For Complex Polynomials

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***Abstract:***
*In this paper we obtain zero-free regions for polynomials with restricted coefficients. These results generalize many already known results on the subject.*
*Mathematics Subject Classification: 30C10, 30C15.*

***Key words**: Coefficient, Polynomial, Zero-free region*

## 1.Introduction And Statement Of Results

In the literature there exist a large number of published papers giving the regions containing some or all the zeros of a polynomial. Recently M. H. Gulzar [2] proved the following results:

*1.1.Theorem A*

Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ and

$$\rho+\alpha_n \ge \alpha_{n-1} \ge ......\ge \alpha_1 \ge \tau\alpha_0,$$

for some $\rho, 0<\tau \le 1$, ,then the number of zeros of P(z) in $|z| \le \frac{R}{k} (R>0, K>0)$, does not exceed

$$\frac{1}{\log k} \log \frac{R^{n+1}[|\rho|+\rho+|\alpha_n|+\alpha_n-\tau(|\alpha_0|+\alpha_0)+2|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R \ge 1$

and

$$\frac{1}{\log k} \log \frac{|a_0|+R[|\rho|+\rho+|\alpha_n|+\alpha_n-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R \le 1$.

*1.2.Theorem B*

Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n with complex coefficients such that for some real $\alpha, \beta$,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, \ j=0,1,2,......,n$$

and

$$|\rho + a_n| \ge |a_{n-1}| \ge ...... \ge |a_1| \ge \tau |a_0| \ ,$$

for some $\rho \ge 0, 0 < \tau \le 1$, then the number of zeros of P(z) in $|z| \le \frac{R}{k}(R > 0, k > 0)$, does not exceed

$$\frac{1}{\log k} \log \frac{(\rho + |a_n|)(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha \sum_{j=1}^{n-1} |a_j| - |a_0|(\cos\alpha - \sin\alpha - 1)}{|a_0|} \ .$$

*1.3.Theorem C*

Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial of degree n such that $\operatorname{Re}(a_j) = \alpha_j$, $\operatorname{Im}(a_j) = \beta_j$ and

$$\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_\lambda, k\alpha_\lambda \ge \alpha_{\lambda-1} \ge ....... \ge \alpha_1 \ge \tau\alpha_0,$$

for $k \ge 1, 0 < \tau \le 1, 0 \le \lambda \le n$, then the number of zeros of P(z) in $|z| \le \frac{R}{k}(R > 0, k > 0)$ does not exceed

$$\frac{1}{\log k} \log \frac{R^{n+1}[|\alpha_n| + \alpha_n + (k-1)(|\alpha_\lambda| + \alpha_\lambda) + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n} |\beta_j|}{|a_0|}$$

for $R \ge 1$

and

$$\frac{1}{\log k} \log \frac{|a_0| + R[|\alpha_n| + \alpha_n + (k-1)(|\alpha_\lambda| + \alpha_\lambda) - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + 2\sum_{j=1}^{n} |\beta_j|}{|a_0|}$$

for $R \le 1$.

In this paper we prove the following results:

*1.4.Theorem 1*

Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\operatorname{Re}(a_j) = \alpha_j$, $\operatorname{Im}(a_j) = \beta_j$ and

$$\rho + \alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_1 \ge \tau\alpha_0 \ ,$$

for some $\rho, 0 < \tau \le 1$, ,then P(z) has no zero in $|z| < \frac{|a_0|}{M_1}$ for $R \ge 1$ and no zero in $|z| < \frac{|a_0|}{M_2}$ for $R \le 1$, where

$$M_1 = R^{n+1}[|\alpha_n| + \alpha_n + |\rho| + \rho - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + 2\sum_{j=0}^{n} |\beta_j|]$$

and

$$M_2 = R[|\alpha_n| + \alpha_n + |\rho| + \rho - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + 2\sum_{j=1}^{n} |\beta_j|].$$

Combining Theorem A and Theorem 1, we get the following result:

*1.5.Corollary 1*

Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ and

$$\rho+\alpha_n \ge \alpha_{n-1} \ge ......\ge \alpha_1 \ge \tau\alpha_0,$$

for some $\rho, 0<\tau\le 1$, ,then the number of zeros of P(z) in $\frac{|a_0|}{M_1}\le |z| \le \frac{R}{k}\,(R>0, K>1)$, does not exceed

$$\frac{1}{\log k}\log\frac{R^{n+1}[|\rho|+\rho+|\alpha_n|+\alpha_n-\tau(|\alpha_0|+\alpha_0)+2|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R\ge 1$

and the number of zeros of P(z) in $\frac{|a_0|}{M_2}\le |z| \le \frac{R}{k}\,(R>0, K>1)$, does not exceed

$$\frac{1}{\log k}\log\frac{|a_0|+R[|\rho|+\rho+|\alpha_n|+\alpha_n-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R\le 1$.

*1.6.Theorem 2*

Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial of degree n such that $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ and

$$\alpha_n \ge \alpha_{n-1} \ge ......\ge \alpha_\lambda, k\alpha_\lambda \ge \alpha_{\lambda-1} \ge .......\ge \alpha_1 \ge \tau\alpha_0,$$

for $k\ge 1, 0<\tau\le 1, 0\le\lambda\le n$, then P(z) has no zero in $|z|<\frac{|a_0|}{M_3}$ for $R\ge 1$ and no zero in $|z|<\frac{|a_0|}{M_4}$ for $R\le 1$, where

$$M_3=R[|\alpha_n|+\alpha_n+(k-1)(|\alpha_\lambda|+\alpha_\lambda)-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+2\sum_{j=1}^{n}|\beta_j|]$$

and

$$M_4=R[|\alpha_n|+\alpha_n+(k-1)(|\alpha_\lambda|+\alpha_\lambda)-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+2\sum_{j=1}^{n}|\beta_j|]$$

Combining Theorem C and Theorem 2, we get the following result:

*1.7.Corollary 2*

Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial of degree n such that $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ and

$$\alpha_n \ge \alpha_{n-1} \ge ......\ge \alpha_\lambda, k\alpha_\lambda \ge \alpha_{\lambda-1} \ge .......\ge \alpha_1 \ge \tau\alpha_0,$$

for $k\ge 1, 0<\tau\le 1, 0\le\lambda\le n$, then the number of zeros of P(z) in $\frac{|a_0|}{M_3}\le |z| \le \frac{R}{k}\,(R>0, k>1)$ does not exceed

$$\frac{1}{\log k}\log\frac{R^{n+1}[|\alpha_n|+\alpha_n+(k-1)(|\alpha_\lambda|+\alpha_\lambda)+2|\alpha_0|-\tau(|\alpha_0|+\alpha_0)+2\sum_{j=0}^{n}|\beta_j|}{|a_0|}$$

for $R\geq 1$

and the number of zeros of P(z) in $\frac{|a_0|}{M_4}\leq|z|\leq\frac{R}{k}(R>0,k>1)$ does not exceed

$$\frac{1}{\log k}\log\frac{|a_0|+R[|\alpha_n|+\alpha_n+(k-1)(|\alpha_\lambda|+\alpha_\lambda)-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+2\sum_{j=1}^{n}|\beta_j|}{|a_0|}$$

for $R\leq 1$.

*1.8.Theorem 3*

Let $P(z)=\sum_{j=0}^{\infty}a_j z^j$ be a polynomial o f degree n with complex coefficients such that for some real $\alpha,\beta$,

$$|\arg a_j-\beta|\leq\alpha\leq\frac{\pi}{2},\ j=0,1,2,......,n$$

and

$$|\rho+a_n|\geq|a_{n-1}|\geq......\geq|a_1|\geq\tau|a_0|,$$

for some $\rho\geq 0, 0<\tau\leq 1$,then P(z) has no zero in $|z|<\frac{|a_0|}{M_5}$ for $R\geq 1$ and no zero in $|z|<\frac{|a_0|}{M_6}$ for $R\leq 1$, where

$$M_5=R^{n+1}[(|\rho|+|a_n|)(\cos\alpha+\sin\alpha+1)-\tau|a_0|(\cos\alpha-\sin\alpha+1)$$
$$+|a_0|+2\sin\alpha\sum_{j=1}^{n-1}|a_j|]$$

and

$$M_6=R[(|\rho|+|a_n|)(\cos\alpha+\sin\alpha+1)-\tau|a_0|(\cos\alpha-\sin\alpha+1)$$
$$+|a_0|+2\sin\alpha\sum_{j=1}^{n-1}|a_j|].$$

Combining Theorem B and Theorem 3, we get the following result:

*1.9.Corollary 3*

Let $P(z)=\sum_{j=0}^{\infty}a_j z^j$ be a polynomial o f degree n with complex coefficients such that for some real $\alpha,\beta$,

$$|\arg a_j-\beta|\leq\alpha\leq\frac{\pi}{2},\ j=0,1,2,......,n$$

and

$$|\rho+a_n|\geq|a_{n-1}|\geq......\geq|a_1|\geq\tau|a_0|,$$

for some $\rho\geq 0, 0<\tau\leq 1$,then the number of zeros of P(z) in $\frac{|a_0|}{M_5}\leq|z|\leq\frac{R}{k}(R>0,k>1)$, does not exceed

$$\frac{1}{\log k}\log\frac{(\rho+|a_n|)(\cos\alpha+\sin\alpha+1)+2\sin\alpha\sum_{j=1}^{n-1}|a_j|-|a_0|(\cos\alpha-\sin\alpha-1)}{|a_0|}$$

**2.Lemma**

For the proof of Theorem 3, we need the following lemma:

*2.1.Lemma*

Let $P(z)=\sum_{j=0}^{\infty}a_j z^j$ be a polynomial o f degree n with complex coefficients such that for some real $\alpha,\beta$, $\left|\arg a_j-\beta\right|\le\alpha\le\frac{\pi}{2}, 0\le j\le n,$ and any t>0, $|ta_j|\ge|a_{j-1}|, 0\le j\le n,$ then,

$$|ta_j-a_{j-1}|\le(t|a_j|-|a_{j-1}|)\cos\alpha+(t|a_j|+|a_{j-1}|)\sin\alpha \text{ .s}$$

Lemma 3 is due to Govil and Rahman [2].

**3.Proofs Of Theorems**

*3.1.Proof Of Theorem 1: Consider The Polynomial*

F(z) =(1-z)P(z)

$$=(1-z)(a_n z^n+a_{n-1}z^{n-1}+......+a_1 z+a_0)$$

$$=-a_n z^{n+1}+(a_n-a_{n-1})z^n+......+(a_1-a_0)z+a_0$$

$$=-a_n z^{n+1}+a_0-\rho z^n+(\rho+\alpha_n-\alpha_{n-1})z^n+......+(\alpha_2-\alpha_1)z^2$$

$$+[(\alpha_1-\tau\alpha_0)+(\tau\alpha_0-\alpha_0)]z+i\sum_{j=0}^{n}(\beta_j-\beta_{j-1})z^j .$$

$$=a_0+G(z)\text{, where}$$

$$G(z)=-a_n z^{n+1}{}_0-\rho z^n+(\rho+\alpha_n-\alpha_{n-1})z^n+......+(\alpha_2-\alpha_1)z^2$$

$$+[(\alpha_1-\tau\alpha_0)+(\tau\alpha_0-\alpha_0)]z+i\sum_{j=0}^{n}(\beta_j-\beta_{j-1})z^j .$$

For $|z|=R$, we have by using the hypothesis,

$$|G(z)|\le|a_n|R^{n+1}+|\rho|R^n+|\rho+\alpha_n-\alpha_{n-1}|R^n+......+|\alpha_2-\alpha_1|R^2+|\alpha_1-\tau\alpha_0|R$$

$$+(1-\tau)|\alpha_0|R+\sum_{j=0}^{n}(|\beta_j|+|\beta_{j-1}|)R^j$$

$$\le R^{n+1}[|\alpha_n|+|\rho|+\rho+\alpha_n-\alpha_{n-1}+......+\alpha_2-\alpha_1+\alpha_1-\tau\alpha_0$$

$$+(1-\tau)|\alpha_0|+|\beta_0|+2\sum_{j=1}^{n}|\beta_j|]$$

$$=R^{n+1}[|\alpha_n|+\alpha_n+|\rho|+\rho-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+2\sum_{j=0}^{n}|\beta_j|]$$

$$=M_1 \quad \text{for } R\ge 1$$

and for $R\le 1$,

$$|G(z)| \le R[|\alpha_n| + \alpha_n + |\rho| + \rho - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + 2\sum_{j=1}^{n}|\beta_j|]$$
$$= M_2$$

Since G(z) is analytic for $|z| \le R$ , G(0)=0, it follows , by Schwarz Lemma that $|G(z)| \le M_1|z|$ for $|z| \le R,\ R \ge 1$ and $|G(z)| \le M_2|z|$ for $|z| \le R,\ R \le 1$.

Therefore, for $|z| \le R,\ R \ge 1$,

$$|F(z)| = |a_0 + G(z)|$$
$$\ge |a_0| - |G(z)|$$
$$\ge |a_0| - M_1|z|$$
$$> 0$$

if $|z| < \dfrac{|a_0|}{M_1}$ ,

and for $|z| \le R,\ R \le 1,\ |F(z)| > 0$ if $|z| < \dfrac{|a_0|}{M_2}$.

This shows that F(z) has no zero in $|z| < \dfrac{|a_0|}{M_1}$ for $R \ge 1$ and no zero in $|z| < \dfrac{|a_0|}{M_2}$ for $R \le 1$.

Since the zeros of P(z) are also the zeros of F(z), the theorem follows.

*3.2.Proof Of Theorem 2: Consider The Polynomial*

F(z) =(1-z)P(z)

$$= (1-z)(a_n z^n + a_{n-1}z^{n-1} + ...... + a_1 z + a_0)$$
$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ...... + (a_1 - a_0)z + a_0$$
$$= -a_n z^{n+1} + a_0 + (\alpha_n - \alpha_{n-1})z^n + ...... + (\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1}$$
$$+ [(k\alpha_\lambda - \alpha_{\lambda-1}) - (k-1)\alpha_\lambda]z^\lambda + (\alpha_{\lambda-1} - \alpha_{\lambda-2})z^{\lambda-1} + ......$$
$$+ [(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)]z + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j$$

$= a_0 + G(z)$, where

$$G(z) = -a_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + ...... + (\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1}$$
$$+ [(k\alpha_\lambda - \alpha_{\lambda-1}) - (k-1)\alpha_\lambda]z^\lambda + (\alpha_{\lambda-1} - \alpha_{\lambda-2})z^{\lambda-1} + ......$$
$$+ [(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)]z + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j$$

For $|z| \le R$, we have by using the hypothesis

$$|G(z)| \le |a_n|R^{n+1} + |\alpha_n - \alpha_{n-1}|R^n + ...... + |\alpha_{\lambda+1} - \alpha_\lambda|R^{\lambda+1} + |k\alpha_\lambda - \alpha_{\lambda-1}|R^\lambda$$
$$+ (k-1)|\alpha_\lambda|R^\lambda + |\alpha_{\lambda-1} - \alpha_{\lambda-2}|R^{\lambda-1} + ...... + |\alpha_1 - \tau\alpha_0|R + (1-\tau)|\alpha_0|R$$
$$+ \sum_{j=1}^{n}(|\beta_j| + |\beta_{j-1}|)R^j$$

$$\leq |a_n|R^{n+1} + R^n[\alpha_n - \alpha_{n-1} + ...... + \alpha_{\lambda+1} - \alpha_\lambda + k\alpha_\lambda - \alpha_{\lambda-1}$$
$$+ (k-1)|\alpha_\lambda| + \alpha_{\lambda-1} - \alpha_{\lambda-2} + ...... + \alpha_1 - \tau\alpha_0 + (1-\tau)|\alpha_0|$$
$$+ |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1}|\beta_j|]$$
$$\leq R^{n+1}[|a_n| + \alpha_n + (k-1)(|\alpha_\lambda| + \alpha_\lambda) - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0|$$
$$+ 2\sum_{j=1}^{n-1}|\beta_j|]$$
$$\leq R^{n+1}[|\alpha_n| + \alpha_n + (k-1)(|\alpha_\lambda| + \alpha_\lambda) - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| \ + 2\sum_{j=1}^{n}|\beta_j|]$$
$$= M_3 \quad \text{for } R \geq 1.$$

and

$$|G(z)| \leq R[|\alpha_n| + \alpha_n + (k-1)(|\alpha_\lambda| + \alpha_\lambda) - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + 2\sum_{j=1}^{n}|\beta_j|]$$
$$= M_4 \quad \text{for } R \leq 1.$$

Since G(z) is analytic for $|z| \leq R$ , G(0)=0, it follows , by Schwarz Lemma that $|G(z)| \leq M_3|z|$ for $|z| \leq R$, $R \geq 1$ and $|G(z)| \leq M_4|z|$ for $|z| \leq R$, $R \leq 1$.

Therefore, for $|z| \leq R$, $R \geq 1$,

$$|F(z)| = |a_0 + G(z)|$$
$$\geq |a_0| - |G(z)|$$
$$\geq |a_0| - M_3|z|$$
$$> 0$$

if $|z| < \dfrac{|a_0|}{M_3}$ ,

and for $|z| \leq R$, $R \leq 1$, $|F(z)| > 0$ if $|z| < \dfrac{|a_0|}{M_4}$.

This shows that F(z) has no zero in $|z| < \dfrac{|a_0|}{M_3}$ for $R \geq 1$ and no zero in $|z| < \dfrac{|a_0|}{M_4}$ for $R \leq 1$.

Since the zeros of P(z) are also the zeros of F(z), the theorem follows.

*3.3.Proof Of Theorem 3: Consider The Polynomial*

F(z) =(1-z)P(z)

$$= (1-z)(a_n z^n + a_{n-1}z^{n-1} + ...... + a_1 z + a_0)$$
$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ...... + (a_1 - a_0)z + a_0$$
$$= -a_n z^{n+1} + a_0 - \rho z^n + (\rho + a_n - a_{n-1})z^n + ...... + (a_2 - a_1)z^2$$
$$+ [(a_1 - \tau a_0) + (\tau a_0 - a_0)]z$$
$$= a_0 + G(z), \text{ where}$$

$$G(z) = -a_n z^{n+1} - \rho z^n + (\rho + a_n - a_{n-1})z^n + ...... + (a_2 - a_1)z^2$$
$$+ [(a_1 - \tau a_0) + (\tau a_0 - a_0)]z .$$

For $|z| \le R$, we have by using the hypothesis and Lemma 3

$$\begin{aligned}
|G(z)| &\le |a_n| R^{n+1} + |\rho| R^n + |\rho + a_n - a_{n-1}| R^n + \ldots\ldots + |a_2 - a_1| R^2 + |a_1 - \tau a_0| R \\
&\quad + (1-\tau)|a_0| R \\
&\le |a_n| R^{n+1} + |\rho| R^n + [(|\rho + a_n| - |a_{n-1}|)\cos\alpha + (|\rho + a_n| + |a_{n-1}|)\sin\alpha] R^n \\
&\quad + \ldots\ldots + [(|a_2| - |a_1|)\cos\alpha + (|a_2| + |a_1|)\sin\alpha] R^2 + (1-\tau)|a_0| R \\
&\quad + [(|a_1| - \tau|a_0|)\cos\alpha + (|a_1| + \tau|a_0|)\sin\alpha] R \\
&\le R^{n+1}[(|\rho| + |a_n|)(\cos\alpha + \sin\alpha + 1) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) \\
&\quad + |a_0| + 2\sin\alpha \sum_{j=1}^{n-1} |a_j|] \\
&= M_5 \quad \text{for } R \ge 1
\end{aligned}$$

and

$$\begin{aligned}
|G(z)| &\le R[(|\rho| + |a_n|)(\cos\alpha + \sin\alpha + 1) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) \\
&\quad + |a_0| + 2\sin\alpha \sum_{j=1}^{n-1} |a_j|] \\
&= M_6 \qquad \text{for } R \le 1.
\end{aligned}$$

Since G(z) is analytic for $|z| \le R$, G(0)=0, it follows , by Schwarz Lemma that $|G(z)| \le M_5 |z|$ for $|z| \le R$, $R \ge 1$ and $|G(z)| \le M_6 |z|$ for $|z| \le R$, $R \le 1$.

Therefore, for $|z| \le R$, $R \ge 1$,

$$\begin{aligned}
|F(z)| &= |a_0 + G(z)| \\
&\ge |a_0| - |G(z)| \\
&\ge |a_0| - M_5 |z| \\
&> 0
\end{aligned}$$

if $|z| < \dfrac{|a_0|}{M_5}$,

and for $|z| \le R$, $R \le 1$, $|F(z)| > 0$ if $|z| < \dfrac{|a_0|}{M_6}$.

This shows that F(z) has no zero in $|z| < \dfrac{|a_0|}{M_5}$ for $R \ge 1$ and no zero in $|z| < \dfrac{|a_0|}{M_6}$ for $R \le 1$.

Since the zeros of P(z) are also the zeros of F(z), the theorem follows.

## 4.References


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