THE INTERNATIONAL JOURNAL OF SCIENCE & TECHNOLEDGE

Two Point Problem for the Abstract Differential Equation of the Second Order

Kudratillo Sadridinovich Fayazov

Professor, Turin Polytechnic University, Tashkent, Uzbekistan

Zamira Shamshaddinovna Abdullaeva

Lecturer, Tashkent University of Information Technologies named after Muhammad al-Kharizmi, Uzbekistan

Abstract:

The work is about the study of the boundary value problem for the second-order differential operator equation. The problem is investigated for conditional correctness. Theorems on uniqueness and conditional stability are proved. An approximate solution of the problem by the quasi-inversion method is constructed, an estimate of the norm between the exact and approximate solutions is obtained, and a formula for the regularization parameter is found.

Keywords: Abstract differential equation, uniqueness, stability, regularization, approximate solution, regularization parameter

1. Introduction

In work [1] the problem was investigated with self-adjoint positive operator coefficient for a homogeneous equation. The multipoint problem for the equation with operator coefficients was studied in work [2], where the conditional correctness of the problem was proved and estimates of conditional stability under other constraints in the operator coefficients were obtained. The multipoint problem has been considered in various aspects by a number of authors (see, work [3], which contains a large number bibliography). The correctness of the multipoint problem in case $n=4$ was showed in work [4]. In work [4], the correctness of the boundary value problem in Banach space investigated, A-linear closed operator, the set of regular points of which contains some angle γ of the complex plane containing the origin of coordinates inside, and the problem of existence and uniqueness are investigated. Works of many authors are devoted to the study of such problems for abstract differential equations, see examples [5], [6] and the literature cited there. In this paper we obtain a priori estimates for the solution of the problem corresponding Hilbert spaces. We prove on the set of correctness theorems characterizing the uniqueness and conditional stability of the solution of the desired problem. Using the quasi-inversion method, we construct an approximate solution, obtain an estimate of the difference between the exact and approximate solution, and obtain a formula for the regularization parameter.

2. Problem Statement

We investigate a problem with data at two points of the regularity domain for solving an abstract second-order differential equation with an operator-type coefficient in Hilbert space.

Let $u(t)$ be a function of the scalar argument t , $0 < t < T$, with values in Hilbert space H . Consider the solution of the differential equation

$$\left(\frac{d}{dt} - A\right)^2 u = f(t), \quad 0 < t < T, \quad (1)$$

satisfying the following conditions:

$$\begin{aligned} u|_{t=0} &= \phi_1, \\ u|_{t=l_1} &= \phi_2, \end{aligned} \quad (2)$$

where A is a self-adjoint operator with a dense domain in H $D(A)$ and $0 < l_1 < T$. The operator A , has a complete orthonormal system of eigen functions $\{\phi_k\}_{k=1}^{\infty}$ and a corresponding system of eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$. Note that the problem (1), (2), generally, is incorrect, namely there is no continuous dependence of the solution on the initial data.

Definition 1. A function $u(t): [0, T] \rightarrow H$ is called a solution of equation (1) if it has a strongly continuous first derivative on segment $[0, T]$, a strong second-order derivative at all interior points of this segment, and satisfies equation [1].

Note that the solutions of problem [1], [2] can be represented as

$$u(t) = v(t) + \eta(t), \quad (3)$$

where $v(t)$ is the solution of (1), (2) with $F, f = 0, v_1(0) = \phi_1, v_2(l_1) = \phi_2$ is the solution of (1) with $f \neq 0, \eta(0) = 0, \eta(l_1) = 0$.

We obtain auxiliary facts for further investigation of the problem.

Lemma 1. (see also [1]). If v_1 and v_2 are the solution of $(\frac{d}{dt} - A)v = 0$ equation, then the $v = v_1 + tv_2$ function is the solution of the problem

$$\begin{aligned} \left(\frac{d}{dt} - A\right)^2 v &= 0, \\ v|_{t=0} &= \phi_1, \\ v|_{t=l_1} &= \phi_2, \end{aligned} \quad (4)$$

and on the contrary, for each given function being the solution of equation (4) there are such functions v_1 and v_2 that

$$v(t) = v_1(t) + tv_2(t). \quad (5)$$

Proof. Let v_1 and v_2 be the solutions of $\left(\frac{d}{dt} - A\right)v_i = 0, i = 1, 2$, respectively, then v is the solution of equation (1).

Indeed

$$\left(\frac{d}{dt} - A\right)v = \left(\frac{d}{dt} - A\right)(v_1 + tv_2) = \left(\frac{d}{dt} - A\right)v_1 + \left(\frac{d}{dt} - A\right)tv_2 = v_2,$$

The result is $\left(\frac{d}{dt} - A\right)v_2 = 0$, which means $\left(\frac{d}{dt} - A\right)\left(\frac{d}{dt} - A\right)v = 0$.

We prove the second part of the Lemma. If the solution of equation (1), it is sufficient to set the choice v_2 .

$$\text{Got } v_2 = \left(\frac{d}{dt} - A\right)v, \quad \text{then } v_1 = v - tv_2.$$

We are to show just $\left(\frac{d}{dt} - A\right)(v - tv_2) = 0$.

As a matter of fact:

$$\begin{aligned} \left(\frac{d}{dt} - A\right)(v - tv_2) &= \left(\frac{d}{dt} - A\right)v - \left(\frac{d}{dt} - A\right)tv_2 = \left(\frac{d}{dt} - A\right)v - v_2 - \left(\frac{d}{dt} - A\right)tv_2 = \\ &= \left(\frac{d}{dt} - A\right)v - \left(\frac{d}{dt} - A\right)v = 0. \end{aligned}$$

It is also proved, that $\left(\frac{d}{dt} - A\right)v_2 = 0$. Lemma is proved.

Lemma 2. To solve the problem (1), (2) the inequality is true

$$\int_0^T \|v(t)\|^2 dt \leq \begin{cases} \int_0^T \|\phi_1\|^{2(1-t/T)} \left(\|v(T)\| + t \left\| \left(\frac{d}{dt} - A \right) v(t) \right\|_{t=T} \right)^{2t/T} dt + \\ + T^2 \int_0^T \left\| \left[\left(\frac{d}{dt} - A \right) v(t) \right]_{t=0} \right\|^{2(1-\frac{t}{l_1})} (\|\phi_1\| + \|\phi_2\|)^{2(\frac{t}{l_1})} \left(\frac{1}{l_1} \right)^{2(\frac{t}{l_1})} dt, & 0 < t < l_1, \\ \int_0^T \|\phi_1\|^{2(1-t/T)} \left(\|v(T)\| + t \left\| \left(\frac{d}{dt} - A \right) v(t) \right\|_{t=T} \right)^{2t/T} dt + \\ + T^2 \int_0^T (\|\phi_1\| + \|\phi_2\|)^{2(\frac{T-t}{T-l_1})} \left(\frac{1}{l_1} \right)^{2(\frac{T-t}{T-l_1})} \left(\left\| \left[\left(\frac{d}{dt} - A \right) v(t) \right]_{t=T} \right\| \right)^{2(\frac{T-t}{T-l_1})} dt, & l_1 < t < T. \end{cases} \quad (6)$$

Proof. By means of representation (5) the solution of a problem (1), (2) it is possible to reduce to the solution of the following problems:

$$v_1 : \begin{cases} \left(\frac{d}{dt} - A \right) v_1 = 0 \\ v_1|_{t=0} = \phi_1, \end{cases} \quad 0 < t < T, \quad (7)$$

$$v_2 : \begin{cases} \left(\frac{d}{dt} - A \right) v_2 = 0 \\ v_2|_{t=l_1} = \frac{\phi_2 - v_1(l_1)}{l_1}, \end{cases} \quad 0 < t < T. \quad (8)$$

For solution of the problem (7) according to [1] and Helder inequality the estimate is valid

$$\begin{aligned} \int_0^T \|v_1(t)\|^2 dt &\leq \int_0^T \|v_1(0)\|^{2(1-t/T)} \|v_1(T)\|^{2t/T} dt \leq \\ &\leq \int_0^T \|\phi_1\|^{2(1-t/T)} \left(\|v(T)\| + t \left\| \left(\frac{d}{dt} - A \right) v(t) \right\|_{t=T} \right)^{2t/T} dt, \quad 0 < t < T. \end{aligned} \quad (9)$$

Similarly [6] for the solution of problem (8) in the case $0 < t < l$ there is an inequality

$$\int_0^T \|v_2(t)\|^2 dt \leq \int_0^T \|v_2(0)\|^{2(1-\frac{t}{l_1})} \|v_2(l_1)\|^{2(\frac{t}{l_1})} dt \leq \int_0^T \left\| \left[\left(\frac{d}{dt} - A \right) v \right]_{t=0} \right\|^{2(1-\frac{t}{l_1})} (\|\phi_1\| + \|\phi_2\|)^{2(\frac{t}{l_1})} \left(\frac{1}{l_1} \right)^{2(\frac{t}{l_1})} dt.$$

Then the following inequality holds for the solution of problem (8)

$$\int_0^T \|v_2(t)\|^2 dt \leq \begin{cases} \int_0^T \left\| \left[\left(\frac{d}{dt} - A \right) v \right]_{t=0} \right\|^{2(1-\frac{t}{l_1})} (\|\phi_1\| + \|\phi_2\|)^{2(\frac{t}{l_1})} \left(\frac{1}{l_1} \right)^{2(\frac{t}{l_1})} dt, & 0 < t < l_1, \\ \int_0^T (\|\phi_1\| + \|\phi_2\|)^{2(\frac{T-t}{T-l_1})} \left(\frac{1}{l_1} \right)^{2(\frac{T-t}{T-l_1})} \left(\left\| \left[\left(\frac{d}{dt} - A \right) v \right]_{t=T} \right\| \right)^{2(\frac{T-t}{T-l_1})} dt, & l_1 < t < T. \end{cases} \quad (10)$$

Summing up the estimates (9) and (10) we find (6).

3. The uniqueness theorem of the problem (4)

Theorem 1. Let A be a self-adjoint operator, then the solution of problem (4) is unique.

Proof. Let there be two solutions to problem (4). Then $v(t) = v_1(t) - v_2(t)$ is the solution of the following problem

$$\left(\frac{d}{dt} - A\right)^2 v = 0, \quad 0 < t < T,$$

$$v|_{t=0} = 0,$$

$$v|_{t=l_1} = 0.$$

It follows from the inequality of Lemma 2 that $\|v(t)\| = 0$ for any $t > 0$ or $v_1(t) \equiv v_2(t)$. Theorem 1 is proved.

$$\text{Let, } M = \left\{ v : \left\| v(T) \right\| + T \left\| \left(\frac{d}{dt} - A \right) v(t) \right\|_{t=T}, \left\| \left(\frac{d}{dt} - A \right) v \right\|_{t=0} \right\| \leq m, \quad m > 0 \right\}.$$

3.1. Theorem of Stability of the Solution Problem (4)

Theorem 2. Let A the self-adjoint positive operator, $v \in M$, and let $\|\phi_1 - \phi_{1\varepsilon}\| \leq \varepsilon$, $\|\phi_2 - \phi_{2\varepsilon}\| \leq \varepsilon$, where $\phi_{1\varepsilon}, \phi_{2\varepsilon}$ is approximate to ϕ_1, ϕ_2 respectively, and v_ε is the solution of problem (4), with $\phi_{1\varepsilon}, \phi_{2\varepsilon}$ data. Then

$$\int_0^T \|V(t)\|^2 dt = \int_0^T \|v(t) - v_\varepsilon(t)\|^2 dt \leq$$

$$\leq \begin{cases} \varepsilon^{2(1-t/T)} (2m + 2Tm)^{2t/T} + 2T^2 m^{2(1-t/l_1)} (2\varepsilon)^{2t/l_1} \left(\frac{1}{l_1}\right)^{2t/l_1}, & 0 < t < l_1, \\ \varepsilon^{2(1-t/T)} (2m + 2Tm)^{2t/T} + T^2 (2\varepsilon)^{2\left(\frac{T-t}{T-l_1}\right)} \left(\frac{1}{l_1}\right)^{2\left(\frac{T-t}{T-l_1}\right)} (2m)^{2\left(\frac{t-l_1}{T-l_1}\right)}, & l_1 < t < T. \end{cases}$$

Proof. Let $v(t)$ solve problem (4) with exact data and $v_\varepsilon(t)$ solve problem (4) with $\phi_{1\varepsilon}, \phi_{2\varepsilon}$ data. Then $V(t) = (v(t) - v_\varepsilon(t))$ satisfies the problem

$$\left(\frac{d}{dt} - A\right)^2 V(t) = 0, \quad 0 < t < T,$$

$$V|_{t=0} = \phi_1 - \phi_{1\varepsilon},$$

$$V|_{t=l_1} = \phi_2 - \phi_{2\varepsilon}. \quad (11)$$

Applying Lemma 1 to problem (11) we have

$$\int_0^T \|V(t)\|^2 dt = \int_0^T \|v(t) - v_\varepsilon(t)\|^2 dt \leq$$

$$\leq \begin{cases} \int_0^T \|\phi_1 - \phi_{1\varepsilon}\|^{2(1-t/T)} \left(\|V(T)\| + t \left\| \left(\frac{d}{dt} - A \right) V(t) \right\|_{t=T} \right)^{2t/T} dt + \\ + T^2 \int_0^T \left\| \left(\frac{d}{dt} - A \right) V(t) \right\|_{t=0}^{2(1-\frac{t}{l_1})} (\|\phi_1 - \phi_{1\varepsilon}\| + \|\phi_2 - \phi_{2\varepsilon}\|)^{2\left(\frac{t}{l_1}\right)} \left(\frac{1}{l_1}\right)^{2\left(\frac{t}{l_1}\right)} dt, & 0 < t < l_1, \\ \int_0^T \|\phi_1 - \phi_{1\varepsilon}\|^{2(1-t/T)} \left(\|V(T)\| + t \left\| \left(\frac{d}{dt} - A \right) V(t) \right\|_{t=T} \right)^{2t/T} dt + \\ + T^2 \int_0^T (\|\phi_1 - \phi_{1\varepsilon}\| + \|\phi_2 - \phi_{2\varepsilon}\|)^{2\left(\frac{T-t}{T-l_1}\right)} \left(\frac{1}{l_1}\right)^{2\left(\frac{T-t}{T-l_1}\right)} \left(\left\| \left(\frac{d}{dt} - A \right) V(t) \right\|_{t=T} \right)^{2\left(\frac{t-l_1}{T-l_1}\right)} dt, & l_1 < t < T. \end{cases}$$

или

$$\int_0^T \|V(t)\|^2 dt = \int_0^T \|v(t) - v_\varepsilon(t)\|^2 dt \leq \begin{cases} \varepsilon^{2(1-t/T)} (2m + 2Tm)^{2t/T} + 2T^2 m^{2(1-t/l_1)} (2\varepsilon)^{2t/l_1} \left(\frac{1}{l_1}\right)^{2t/l_1}, & 0 < t < l_1, \\ \varepsilon^{2(1-t/T)} (2m + 2Tm)^{2t/T} + T^2 (2\varepsilon)^{2\left(\frac{T-t}{T-l_1}\right)} \left(\frac{1}{l_1}\right)^{2\left(\frac{T-t}{T-l_1}\right)} (2m)^{2\left(\frac{t-l_1}{T-l_1}\right)}, & l_1 < t < T. \end{cases}$$

Theorem 2 is proved.

By means of representation (3) the solution of a problem (1), (2) it is possible to reduce to the solution of the following problems:

$$\begin{aligned} \left(\frac{d}{dt} - A\right)^2 \eta &= f, \\ \eta|_{t=0} &= 0, \\ \eta|_{t=l_1} &= 0, \end{aligned} \quad (12)$$

you can convert task (12) to the following task

$$\begin{aligned} \left(\frac{d}{dt} - A\right)^2 \eta &= f, \\ \eta|_{t=0} &= 0, \\ \eta'|_{t=0} &= \psi. \end{aligned} \quad (13)$$

$$\text{where } \psi(l) = -\frac{1}{l_1} \int_0^{l_1} e^{-A\tau} (l_1 - \tau) f(\tau) d\tau.$$

Lemma 3. The solution of problem (12) is equivalent to the solution of problem (13).

Proof. Let in (12) A the self-adjoint operator have corresponding eigenvalues and eigen functions. Then using the

expansions $\eta(t) = \sum_{k=0}^{\infty} \eta_k \varphi_k$ we have

$$\left(\frac{d}{dt} - \lambda\right)^2 \eta_k = f_k,$$

where $f_k = (f, \varphi_k)$, $k = 1, 2, \dots$, $\eta_k|_{t=0, t=l_1} = 0$.

Let $\eta_k(t)$ be the solution of the problem (12). Then $\eta_k(t)$ is represented as:

$$\eta_k(t) = C_1 e^{\lambda t} + t C_2 e^{\lambda t} + \int_0^t e^{\lambda(t-\tau)} (t-\tau) f_k(\tau) d\tau. \quad (14)$$

Suppose $\eta_k|_{t=0, t=l_1} = 0$, and $f_k(t) = 0$, then it is obvious that $\eta_k(t) \equiv 0$.

Under this condition, we prove that problem (13) also has zero value. Since $f_k = 0$, then and $\psi_k = 0$ then the well-known problem (13) has a zero solution, i.e. all $\eta_k(t) = 0$. Since by its coefficient the function $\eta(t) = \sum_{k=0}^{\infty} \eta_k \varphi_k$ is defined uniquely then $\eta \equiv 0$.

4. A priori Estimate for Problem (1), (2)

Consider problem (13) and introduce $w(t) = \frac{d\eta}{dt} - A\eta$ notation, then

$$\begin{aligned} \frac{dw}{dt} - Aw &= f(t), \\ w(0) &= \psi, \end{aligned} \quad (17)$$

and

$$\begin{cases} \frac{d\eta}{dt} - A\eta = w(t), \\ \eta(0) = 0. \end{cases} \quad (18)$$

$$\text{Denote by } M = \left\{ \left\| \left[\left(\frac{d}{dt} - A \right) \eta \right]_{t=T} \right\| + \left(\int_0^T \|f(\tau)\|_0^2 d\tau \right)^{1/2} \leq m, \quad m > 0 \right\}.$$

There is a Lemma.

Lemma 4. Let A be a self-adjoint operator and $\eta(0) = 0$, $\eta'(0) = \psi$. Then for the solution of problem (13) is true the following estimate

$$\|\eta(t)\|_0 \leq \sqrt{2} (\beta)^{1-\frac{t}{T}} \left(\|\eta(T)\|_0 + \beta \right)^{\frac{t}{T}} + \beta, \quad (19)$$

$$\text{where } \beta = \left(\int_0^T \left(\sqrt{2} \left(\|\psi\| + \left(\int_0^T \|f(\tau)\|_0^2 d\tau \right)^{1/2} \right)^{1-\frac{t}{T}} (m)^{\frac{t}{T}} + \left(\int_0^T \|f(\tau)\|_0^2 d\tau \right)^{1/2} \right)^2 d\tau \right)^{1/2}.$$

Proof. The solution of the operator equation (17) can be represented as

$$w(t) = w_1(t) + w_2(t),$$

where $w_1(t)$ is the solution of the inhomogeneous equation (17) and $w_2(t)$ is the solution of the homogeneous equation (17), and holds

$$\eta(t) = \sum_{k=1}^{\infty} (\eta_k^+(t) \varphi_k^+ + \eta_k^-(t) \varphi_k^-), \quad w(t) = \sum_{k=1}^{\infty} (w_k^+(t) \varphi_k^+ + w_k^-(t) \varphi_k^-),$$

where

$$\begin{aligned} \lambda > 0, \quad w_{1k}^+(t) &= -\int_t^T e^{\lambda_k^+(t-\tau)} f_k^+(\tau) d\tau, \quad w_{2k}^+(t) = \int_0^T e^{\lambda_k^+(t-\tau)} f_k^+(\tau) d\tau, \\ \lambda < 0, \quad w_{1k}^-(t) &= \int_0^t e^{\lambda_k^-(t-\tau)} f_k^-(\tau) d\tau, \quad w_{2k}^-(t) = 0. \end{aligned}$$

According to the definition of the norm for the function $w_1(t)$ the inequality is true

$$\|w_1(t)\|_0^2 \leq \sum_{k=1}^{\infty} \left(|w_{1k}^+(t)|^2 + |w_{1k}^-(t)|^2 \right) \leq \int_0^T \|f(\tau)\|_0^2 d\tau. \quad (20)$$

For function $w_2(t)$, the estimate [7]

$$\|w_2(t)\|_0 \leq \sqrt{2} \left(\|w_2(0)\|_0 \right)^{1-\frac{t}{T}} \left(\|w_2(T)\|_0 \right)^{\frac{t}{T}}, \quad 0 < t < T. \quad (21)$$

is valid. Since $w_2(t) = w(t) - w_1(t)$ then from inequalities (20), (21) follows

$$\begin{aligned} \|w(t)\|_0 &\leq \sqrt{2} \left(\|w(0)\|_0 + \|w_1(0)\|_0 \right)^{1-\frac{t}{T}} \left(\|w(T)\|_0 + \|w_1(T)\|_0 \right)^{\frac{t}{T}} + \|w_1(t)\|_0. \\ \|w(t)\|_0 &\leq \sqrt{2} \left(\|\psi\| + \left(\int_0^T \|f(\tau)\|_0^2 d\tau \right)^{1/2} \right)^{1-\frac{t}{T}} \left(\|w(T)\|_0 + \left(\int_0^T \|f(\tau)\|_0^2 d\tau \right)^{1/2} \right)^{\frac{t}{T}} + \left(\int_0^T \|f(\tau)\|_0^2 d\tau \right)^{1/2}, \end{aligned}$$

Similarly, the solution of the problem (18) is estimated and we have

$$\begin{aligned} \|\eta(t)\|_0 &\leq \sqrt{2} \left(\|\eta(0)\|_0 + \|\eta_1(0)\|_0 \right)^{1-\frac{t}{T}} \left(\|\eta(T)\|_0 + \|\eta_1(T)\|_0 \right)^{\frac{t}{T}} + \|\eta_1(t)\|_0, \\ \|\eta(t)\|_0 &\leq \sqrt{2} (\beta)^{1-\frac{t}{T}} \left(\|\eta(T)\|_0 + \beta \right)^{\frac{t}{T}} + \beta, \end{aligned}$$

$$\text{where } \beta = \left(\int_0^T \left(\sqrt{2} \left(\|v\| + \left(\int_0^T \|f(\tau)\|_0^2 d\tau \right)^{1/2} \right)^{1-\frac{t}{T}} \left(\left\| \left[\left(\frac{d}{dt} - A \right) \eta \right]_{t=T} \right\| + \left(\int_0^T \|f(\tau)\|_0^2 d\tau \right)^{1/2} \right)^{\frac{t}{T}} + \left(\int_0^T \|f(\tau)\|_0^2 d\tau \right)^{1/2} \right)^2 d\tau \right)^{1/2}.$$

4.1. Uniqueness and Stability Theorems of the Problem (13)

Theorem 3. Let be A -self-adjoint operator, then the solution of problem (13) is unique.

Proof. Suppose there are two different solutions η_1 and η_2 . Then their difference $\eta = \eta_1 - \eta_2$ is the solution of the following Cauchy problem

$$\left(\frac{d}{dt} - A \right)^2 \eta = 0,$$

$$\eta|_{t=0} = 0,$$

$$\eta'|_{t=0} = 0.$$

Acting in exactly the same way as in Lemma 4, from the estimate (19), it follows that $\int_0^T \|\eta\|^2 d\tau = 0$. From $\eta \equiv 0$ i.e.

$$\eta_1 = \eta_2.$$

Theorem 4. Let the solution of problem (13) exist and $\eta \in M$ then the estimate (19) is correct.

The proof of theorems 4 follows directly from Lemma 4 and estimate (19).

5. Approximate Solution of the Problem by the Quasi-Inversion Method

Let A be a self-adjoint operator having a complete orthonormal system of eigenfunctions $\{\xi_k\}_{k=1}^\infty$ and a corresponding system of eigenvalues $\{\lambda_k\}_{k=1}^\infty$ arranged in ascending order: $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$.

Consider some special cases of problem (1), (2).

A) Let the problem (1), (2) $\phi_2 = 0$, $f = 0$, $u \in M$.

Then the solution of the problem (1), (2) can be represented as

$$u(t) = \sum_{k=1}^\infty u_k(t) \xi_k(t),$$

$$\text{где } u_k(t) = \left(1 - \frac{t}{l_1} \right) e^{k^2 t} \phi_{1k}, \quad \phi_{1k} = (\phi_1, \xi_k), \quad k = 1, 2, \dots$$

Auxiliary problem 1. (see also [8]). Consider the solution of a differential equation

$$\left(\frac{d}{dt} - k^2(1 - \alpha k^2) \right)^2 u_{\alpha k} = 0,$$

$$u_{\alpha k}|_{t=0} = \phi_1,$$

$$u_{\alpha k}|_{t=l_1} = 0, \quad \alpha > 0, \quad k = 1, 2, \dots,$$

and its solution has the form

$$u_{\alpha}(t) = \sum_{k=1}^\infty u_{\alpha k}(t) \xi_k(t),$$

$$\text{где } u_{\alpha k}(t) = \left(1 - \frac{t}{l_1} \right) e^{k^2 t - \alpha k^4 t} \phi_{1k}, \quad \phi_{1k} = (\phi_1, \xi_k), \quad k = 1, 2, \dots$$

Consider the norm of difference between the exact and approximate solution

$$\begin{aligned} \|u(t) - u^{(\alpha\varepsilon)}(t)\| &= \|u(t) - u_{\alpha}(t) + u_{\alpha}(t) - u^{(\alpha\varepsilon)}(t)\| \leq \\ &\leq \|u(t) - u_{\alpha}(t)\| + \|u_{\alpha}(t) - u^{(\alpha\varepsilon)}(t)\|. \end{aligned} \quad (22)$$

Here triangle inequality is used for norms.

Now we estimate, given $\|u(T)\|^2 \leq \left(1 - \frac{T}{l_1}\right)^2 \sum_{k=1}^{\infty} e^{2k^2 T} \phi_{1k}^2 \leq m^2$, the first term in the right side of the inequality (22)

$$\|u(t) - u_{\alpha}(t)\|^2 \leq \left(1 - \frac{t}{l_1}\right)^2 \sum_{k=1}^{\infty} \phi_{1k}^2 (e^{k^2 t} - e^{k^2(1-\alpha k^2)t})^2, \quad (23)$$

$$\text{use condition } \phi_{1k}^2 \leq \left(1 - \frac{T}{l_1}\right)^{-2} e^{-2k^2 T} m^2.$$

To determine the conditional extremum we use the Lagrange method

$$\|u(t) - u_{\alpha}(t)\| \leq \left(\frac{l_1 - t}{l_1 - T}\right) (e^{k^2 t} - e^{k^2(1-\alpha k^2)t}) m e^{-k^2 T} \leq \left(\frac{l_1 - t}{l_1 - T}\right) e^{-k^2(T-t)} \alpha k^4 t m = \beta(k).$$

The extremum of the function $\beta(k)$ is related to the transcendental equation $\beta'(k) = 0$. We find an estimate of the extremum coinciding in α order, differentiating the function $\beta(k)$ we have

$$\beta'(k) = \left(\frac{l_1 - t}{l_1 - T}\right) m (-2k(T-t) e^{-k^2(T-t)} \alpha k^4 t + e^{-k^2(T-t)} 4\alpha k^3 t) = 0,$$

$$\text{find } k = \sqrt{\frac{2}{T-t}}, \text{ and finally,}$$

$$\|u(t) - u_{\alpha}(t)\| \leq \left(\frac{l_1 - t}{l_1 - T}\right) e^{-2} \times \frac{4\alpha t m}{(T-t)^2}. \quad (24)$$

Let the solution of problem (1), (2) exist, $\|\phi_1 - \phi_{1\varepsilon}\| < \varepsilon$ and $u \in M$.

As an approximate solution of the problem (1), (2) with approximate data we can consider the function

$$u^{(\alpha\varepsilon)}(t) = \sum_{k=1}^{\infty} u_{\alpha k\varepsilon}(t) \xi_k(t),$$

$$\text{where } K \text{ is a regularization parameter, and } u_{\alpha k\varepsilon}(t) = \left(1 - \frac{t}{l_1}\right) e^{k^2 t - \alpha k^4 t} \phi_{1k\varepsilon}, \quad \phi_{1k\varepsilon} = (\phi_{1\varepsilon}, \xi_k), \quad k = 1, 2, \dots$$

We estimate the second part of the inequality in (22)

$$\begin{aligned} \|u_{\alpha}(t) - u^{(\alpha\varepsilon)}(t)\|^2 &\leq \left(1 - \frac{t}{l_1}\right)^2 \sum_{k=1}^{\infty} (e^{k^2(1-\alpha k^2)t})^2 (\phi_{1k} - \phi_{1k\varepsilon})^2 \leq \\ &\leq \left(1 - \frac{t}{l_1}\right)^2 \max_k e^{2(k^2 - \alpha k^4)t} \sum_{k=1}^{\infty} (\phi_{1k} - \phi_{1k\varepsilon})^2 \leq \left(1 - \frac{t}{l_1}\right)^2 \varepsilon^2 e^{\frac{t}{4\alpha}}, \end{aligned}$$

$$\text{consequently } \|u_{\alpha}(t) - u^{(K\varepsilon)}(t)\| \leq \left(1 - \frac{t}{l_1}\right) e^{\frac{t}{4\alpha}} \varepsilon.$$

Finally we have

$$\|u(t) - u^{(\alpha\varepsilon)}(t)\| \leq \frac{l_1 - t}{l_1 - T} e^{-2} \frac{4\alpha t m}{(T-t)^2} + \left(1 - \frac{t}{l_1}\right) e^{\frac{t}{4\alpha}} \varepsilon.$$

Б) let in the problem (1), (2) $\phi_1 = 0$, $f = 0$, $u \in M$.

Then the solution of the problem (1), (2) can be represented as

$$u(t) = \sum_{k=1}^{\infty} u_k(t) \xi_k(t),$$

$$\text{где } u_k(t) = \frac{t}{l_1} e^{k^2(t-l_1)} \phi_{2k}, \quad \phi_{2k} = (\phi_2, \xi_k), \quad k = 1, 2, \dots$$

Auxiliary problem 2. Consider the solution of a differential equation

$$\left(\frac{d}{dt} - k^2(1 - \alpha k^2)\right)^2 u_{\alpha k} = 0,$$

$$u_{\alpha k}|_{t=0} = 0,$$

$$u_{\alpha k}|_{t=l_1} = \phi_2, \quad \alpha > 0.$$

This problem is correct and its solution has the form

$$u_{\alpha}(t) = \sum_{k=1}^{\infty} u_{\alpha k}(t) \xi_k(t),$$

$$\text{где } u_{\alpha k}(t) = \frac{t}{l_1} e^{(k^2 - \alpha k^4)(t-l_1)} \phi_{2k}, \quad \phi_{2k} = (\phi_2, \xi_k), \quad k = 1, 2, \dots$$

Consider the norm of difference between the exact and approximate solution

$$\begin{aligned} \|u(t) - u^{(K\varepsilon)}(t)\| &= \|u(t) - u_{\alpha}(t) + u_{\alpha}(t) - u^{(K\varepsilon)}(t)\| \leq \\ &\leq \|u(t) - u_{\alpha}(t)\| + \|u_{\alpha}(t) - u^{(K\varepsilon)}(t)\|. \end{aligned} \quad (25)$$

Now we estimate under condition $\|u(T)\|^2 \leq \left(\frac{T}{l_1}\right)^2 \sum_{k=1}^{\infty} e^{2k^2(T-l_1)} \phi_{2k}^2 \leq m^2$ the first term in the right part of the inequality (25)

$$\|u(t) - u_{\alpha}(t)\|^2 \leq \left(\frac{t}{l_1}\right)^2 \sum_{k=1}^{\infty} (e^{k^2(t-l_1)} - e^{(k^2 - \alpha k^4)(t-l_1)})^2 \phi_{2k}^2,$$

$$\phi_{2k}^2 \leq m^2 e^{-2k^2(T-l_1)} \left(\frac{T}{l_1}\right)^{-2} \text{ condition is used}$$

To determine the conditional extremum we use the Lagrange method

$$\|u(t) - u_{\alpha}(t)\| \leq \left(\frac{t}{T}\right) (e^{k^2(t-l_1)} - e^{(k^2 - \alpha k^4)(t-l_1)}) m e^{-k^2(T-l_1)} \leq \left(\frac{t}{T}\right) e^{k^2(t-T)} \alpha k^4 (t-l_1) m = \gamma(k).$$

The extremum of the function $\gamma(k)$ is related to the transcendental equation $\gamma'(k) = 0$. We find the estimate of the extremum coinciding in α order, differentiating the function $\gamma(k)$ we have

$$\gamma'(k) = \left(\frac{t}{T}\right) m \left(-2k(T-t) e^{-k^2(T-t)} \alpha k^4 (t-l_1) + e^{-k^2(T-t)} 4\alpha k^3 (t-l_1)\right) = 0, \quad \text{или} \quad k = \sqrt{\frac{2}{T-t}},$$

such as

$$\|u(t) - u_{\alpha}(t)\| \leq \frac{t}{T} e^{-2} \times \frac{4\alpha(t-l_1)m}{(T-t)^2}.$$

Let the solution of problem (1), (2) exist, $\|\phi_2 - \phi_{2\varepsilon}\| < \varepsilon$ and $u \in M$.

As an approximate solution of the problem (1), (2) with approximate data we can consider the function

$$u^{(K\varepsilon)}(t) = \sum_{k=1}^{\infty} u_{\alpha k\varepsilon}(t) \xi_k(t),$$

where K is a regularization parameter, and $u_{\alpha k\varepsilon}(t) = \frac{t}{l_1} e^{(k^2 - \alpha k^4)(t-l_1)} \phi_{2k\varepsilon}$, $\phi_{2k\varepsilon} = (\phi_{2\varepsilon}, \xi_k)$, $k = 1, 2, \dots$ Now we estimate the second part of the inequality in (25)

$$\|u_{\alpha}(t) - u^{(K\varepsilon)}(t)\|^2 \leq \left(\frac{t}{l_1}\right)^2 \sum_{k=1}^{\infty} (e^{(k^2 - \alpha k^4)(t-l_1)})^2 (\phi_{2k} - \phi_{2k\varepsilon})^2 \leq$$

$$\leq \left(\frac{t}{l_1}\right)^2 \max_k e^{2(k^2 - \alpha k^4)(t-l_1)} \sum_{k=1}^{\infty} (\phi_{2k} - \phi_{2k\varepsilon})^2 \leq \left(\frac{t}{l_1}\right)^2 \varepsilon^2 e^{\frac{2(t-l_1)}{4\alpha}}$$

or

$$\|u_{\alpha}(t) - u^{(K\varepsilon)}(t)\| \leq \frac{t}{l_1} e^{\frac{(t-l_1)}{4\alpha}} \varepsilon.$$

Finally we have

$$\|u(t) - u^{(K\varepsilon)}(t)\| \leq \frac{t}{T} e^{-2} \times \frac{4\alpha(t-l_1)m}{(T-t)^2} + \frac{t}{l_1} e^{\frac{(t-l_1)}{4\alpha}} \varepsilon.$$

By minimizing the right-hand side of the latter inequality in ε , for $\varepsilon > 0$ we find a formula for the regularization parameter K .

6. Conclusion

In recent years, due to the rapid development of computing technologies and computer science, the theory of inverse ill-posed problems is one of the important areas of the equation of mathematical physics and computational mathematics. In this paper we study the internal boundary value problem for the differential operator equation of the 2nd order. The question of constructing an estimate of the approximate solution of the problem is considered. The theorem on uniqueness and conditional stability for the solution of the considered problem on the corresponding set of correctness is proved.

7. References

- i. Atakhodzhaev M. A. Egamberdiyev O. M. (1990) Cauchy problem for abstract second order equation. Siberian mathematical journal. V. 31, No. 4., pp. 187-191.
- ii. Valitsky Yu. N. (1988) Correctness of the multipoint problem for the equation with the operator coefficients, V.4, pp. 44-53.
- iii. Ptashnyk B. I. (1984). Ill-posed boundary value problems for partial differential equations. Kyiv: Sciences, Dumka, 264 p.
- iv. Valitsky Yu. N. (1981). A four-point problem for a differential equation in Banach space. Functional analysis and its applications, Vol. 15, issue 4, pp. 69-70.
- v. Alaminov M. H. (2002). The Cauchy problem for a high-order operator-differential equation. Bulletin of KOANRUZ. No. 1-2, pp. 54-57.
- vi. Lavrentev M. M., Savelev L. Ya., (1991). Linear operators and ill-posed problems. M.: Science, 327 p.
- vii. Fayazov K. S., Khudayberganov J. K. (2018). An ill-posed boundary value problem for a mixed-type equation with two degeneracy lines. Uzbek Mathematical Journal, No. 2, pp 32-42; //doi:10.29229/uzmj.2018-2-3.
- viii. Fayazov K. S. (2001) Methods of solution ill-posed problem of numerical mathematics, mathematical physics and analysis. Tashkent, NUUZ, pp. 82-90.
- ix. Zinovii N, Volodymyr I, Petro P, Oksana M. (2019). The differential-symbol method of constructing the quasipolynomial solutions of two-point in time problem for nonhomogeneous partial differential equation, Turkish Journal of Mathematics, 43: 1241 – 1252, //doi:10.3906/mat-1901-17.
- x. Baskakov A. G., Kabantsova L. Yu., Kostrub I. D., Smagina T. I. (2017). Linear Differential Operators and Operator Matrices of the Second Order, Differential equations, Vol. 53, No. 1, pp. 10-19.
- xi. Kalenyuk P. I., Kuduk G., Kohut V., Nytrebych Z. M. (2015). Journal of Mathematical Sciences, Vol. 208, No. 3, July, pp. 267-272. // doi: 10.1007/s10958-015-2444-z.